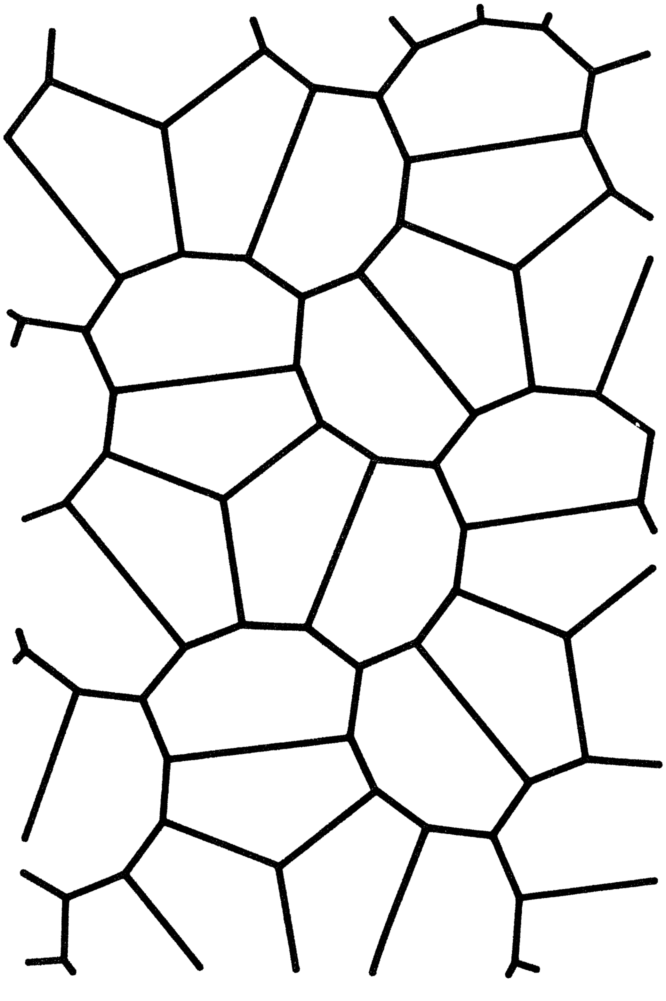
MATHEMATICS MAGAZINE



Vol. 60 No. 2
April 1987

TILINGS • THE CHEAPEST SANDBOX

MAORI GAMES • SINES AND COSINES

MATHEMATICS MAGAZINE

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ARTICLES

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PROBLEMS

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EDITORIAL POLICY

The aim of *Mathematics Magazine* is to provide lively and appealing mathematical exposition. This is not a research journal and, in general, the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for an article for the *Magazine*. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Articles on pedagogy alone, unaccompanied by interesting mathematics, are not suitable. Neither are articles consisting mainly of computer programs unless these are essential to the presentation of some good mathematics. Manuscripts on history are especially welcome, as are those showing relationships between various branches of mathematics and between mathematics and other disciplines.

In addition to articles and notes the *Magazine* solicits proofs without words, mathematical verse, anecdotes, cartoons, and other such material consistent with the level and aims described above. Letters and comments are also welcome.

The full statement of editorial policy appears in this *Magazine*, Vol. 54, pp. 44–45, and is available from the Editor. Manuscripts to be submitted should not be concurrently submitted to, accepted for publication by, nor published by another journal or publisher.

Send new manuscripts to: G. L. Alexanderson, Editor, Mathematics Magazine, Santa Clara University, Santa Clara, CA 95053. Manuscripts should be typewritten and double spaced and prepared in a style consistent with the format of *Mathematics Magazine*. Authors should submit the original and one copy and keep one copy. Illustrations should be carefully prepared on separate sheets in black ink, the original without lettering and two copies with lettering added.

AUTHORS

Ludwig Danzer, **Branko Grünbaum**, and **G. C. Shephard** share an interest in all things geometric, besides being individually attracted to number theory, combinatorics, group theory, and other topics. They have collaborated, among themselves and with others, in various combinations and for many years (the first joint paper by two of them appeared in 1962). During the last ten years they have devoted most of their energies to studying problems on tilings. They see this field as particularly suitable for attracting people to geometry, since many of its results and problems are easily explained and understood while being far from trivial. The development of the material presented here started in 1982, during visits of B.G. to Dortmund and Norwich—the geographic separation of the authors contributed to the long period of gestation.

Marcia Ascher, Professor of Mathematics at Ithaca College, received degrees from Queens College (CUNY) and UCLA. She has done work in numerical analysis and applications of mathematics to archeology. Her extensive study of the logical-numerical system of Inca quipus led to her major current interest in ethnomathematics, the mathematical ideas of nonliterate peoples.

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Equitransitive Tilings, or How to Discover New Mathematics*

LUDWIG DANZER
Universität Dortmund
D 46 Dortmund 50, Federal Republic of Germany

BRANKO GRÜNBAUM
University of Washington
Seattle, WA 98195

G. C. SHEPHARD
University of East Anglia
Norwich NR4 7TJ, England

1. Introduction

Every mathematician must have had the experience of being asked "How does one do research in mathematics?" and this question is often accompanied by some remark to the effect that in a subject with such a long history "Hasn't everything been discovered by now?" In this note we describe an example which can help to answer the first question while refuting the implication of the second.

To illustrate our comments we shall consider a specific problem concerned with tiling the plane by convex polygons. We have chosen this for several reasons: the problem is easy to understand, it has considerable visual appeal, and it can be used pedagogically to introduce students at all levels to important, but often neglected, aspects of geometry, topology, and combinatorics. It shows that there still exist tractable problems which are open in spite of the fact that they could have been understood (and even solved!) by geometers in ancient times.

The crux of any research project is, of course, the "problem." There is not the slightest difficulty in finding unsolved problems—the literature abounds with these—but to find an interesting and relevant problem for which there is a reasonable possibility of solution is a somewhat tougher proposition. Perhaps the best approach is not to begin with a predetermined problem but to select a field of study. For example, here as an illustration we shall investigate *transitivity questions concerning plane tilings by convex tiles*. (For those readers to whom some of the words in the previous sentence are unfamiliar, explanations will follow shortly.) We must begin by finding out what is known. Fortunately, in this case the literature is not very extensive, and one can discover recent material by searching *Mathematical Reviews*, or better, by enquiring from someone who already works in the field. In any case, if a well-written, recent survey paper exists, it is invaluable, since its bibliography should report on all the latest work in the subject. One surprising fact is very important from our point of view: even in a subject as elementary as plane Euclidean geometry, an astonishing amount is *not* known, so there is considerable scope for research.

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Let us be more specific. For our illustration we shall consider **edge-to-edge tilings** of the **Euclidean plane** in which each **tile** is a **convex** polygon. (For the convenience of the reader, definitions of words printed in heavy type are collected in Appendix I. All the results discussed in the article, as well as their proofs, remain valid for much more general tilings, the "normal tilings" discussed in [**8**], [**10**]; the restrictions imposed here are for convenience only, in order to make the paper more easily readable.) Of particular interest is the **symmetry group** of such a tiling; we say that two tiles belong to the same **transitivity class** if one can be mapped onto the other by a symmetry of the tiling. Clearly, if two tiles belong to the same transitivity class then they must be congruent, but the converse is not true, as can be seen from an example like that shown in FIGURE 1. Let us consider some of the interesting mathematical problems that arise naturally out of the concept of transitivity.

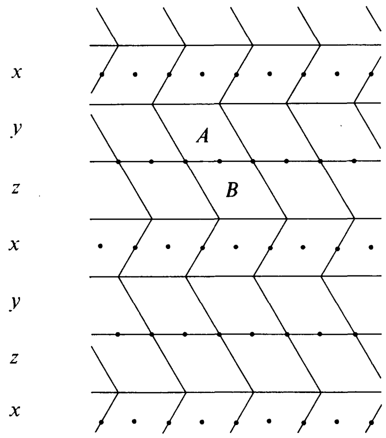
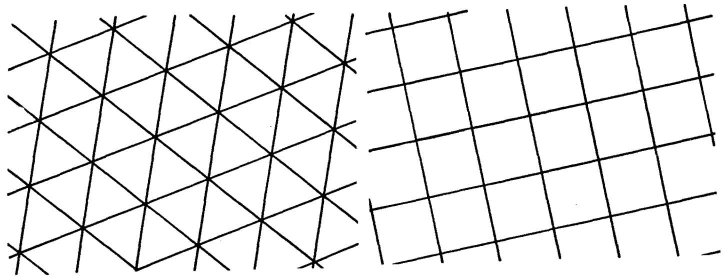
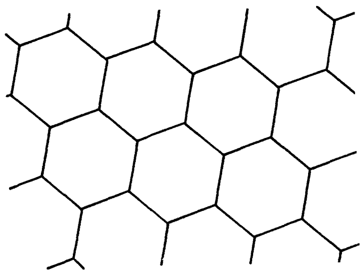


FIGURE 1. An example of a tiling by congruent rhombs. (Here, and in the other diagrams, we show only a small "patch" of the tiling which is supposed to extend repetitively over the whole plane.) There are two transitivity classes of tiles, one consisting of the rhombs in the rows marked x, and the other of the rhombs in the rows marked y and z. Any rhomb can be mapped by a translation onto any other rhomb in the same row or in any other row that is marked with the same letter. Moreover, any such translation is a symmetry of the tiling. Hence all the rhombs in the x-rows belong to the same transitivity class. The same is true of rhombs in the y-rows and z-rows, but these form a single transitivity class since any rhomb in a y-row can be mapped onto any rhomb in a z-row by a half-turn about one of the points marked by a dot in the diagram. It is easily verified that each of these half-turns is a symmetry of the tiling. To complete the proof of the assertion that there are just two transitivity classes we need to show that no symmetry of the tiling will map a rhomb in an x-row onto any rhomb in a y-row or z-row. This can be done either by considering the set of *all* symmetries of the tiling (namely, the translations and half-turns mentioned above and their products), or by remarking that the tiles surrounding a rhomb in an x-row are arranged differently from those surrounding a rhomb in a y-row or a z-row.

Are there any tilings with just *one* transitivity class of tiles? In other words, are there tilings in which each tile can be mapped onto any other tile by some symmetry of the tiling? If, moreover, we insist that the tiles be *regular* polygons, then there are just three possibilities, namely, the familiar "regular tilings" which have been known since ancient times; see FIGURE 2. That there are only three regular tilings follows from the well-known value $(n-2)180°/n$ for the angle at a



(3^6) (4^4)



(6^3)

FIGURE 2. **The three regular tilings. These are the only tilings in which the tiles are regular polygons and all tiles belong to one transitivity class. Note that we do not consider as different such tilings which can be brought into coincidence by any rigid motion, or by change of scale (similarity). Here, and in other illustrations of tilings by regular polygons, we indicate the "symbol" of the tiling—that is, for each of the different kinds of vertices, we list the cyclic sequence of polygons around a vertex.**

corner of a regular n-gon. This is a factor of 360° if and only if $n = 3$, 4, or 6, and hence only these values can lead to regular tilings. (It is worth remarking, however, that there are infinitely many tilings in which the tiles are regular polygons if a mixture of kinds of tiles is allowed, that is, if we may use n-gons for several different values of n; see [**5**]). If we do not insist on regularity of the (polygonal and convex) tiles, then tilings by n-gons in which all tiles belong to the same transitivity class are possible for $n = 3$, 4, 5, or 6 (but not for $n \geq 7$; see [**10**] or [**14**]). Moreover, each such tiling belongs to one of 47 types; examples of some of them are shown in FIGURE 3. (Diagrams of all these types, as well as of types which are not edge-to-edge, can be found in [**6**].) It would lead us too far from our subject to discuss the exact meaning of the word "type" in this context; suffice it to say that it is well defined and more or less coincides with our intuitive idea of its meaning. (For a full discussion of the classification of tilings into "types" see [**4**], [**7**], [**9**].) We note that well over 2000 years separate the dates at which the first and the last of the enumerations of tilings mentioned so far have been carried out!

The above facts introduce us to a whole class of *enumerative problems* which are easy to state, but not so easy to solve. For example, suppose we ask now for tilings in which the tiles belong to exactly two transitivity classes. Restricting attention to regular polygonal tiles, it was found (see [**2**]) by I. Debroey and L. Landuyt (who were, at that time, graduate students at the Free University of Brussels, Belgium) that there are exactly 13 types of such tilings; these are shown in

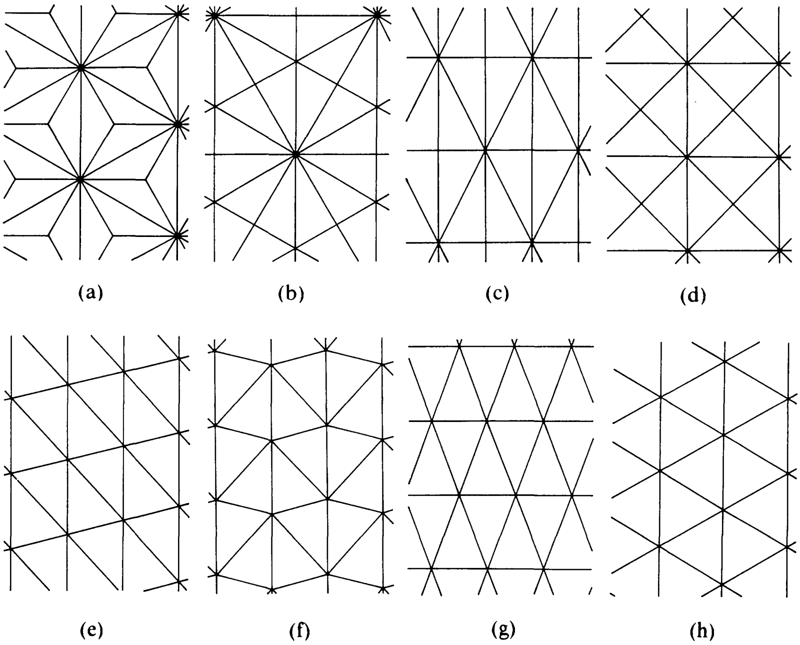


FIGURE 3. **The eight types (which are all the types possible according to one classification method (see [6]) of edge-to-edge tilings by triangles in which all the tiles belong to one transitivity class. These tilings are of four topological types; see Appendix I.**

FIGURE 4. However, if we drop the condition of regularity and restrict the tilings only by requiring that the tiles are convex polygons which belong to two transitivity classes, the situation changes drastically. It is known (see [3]) that there are at least 503 *topologically distinct* types of such tilings. As examples, in FIGURE 5 we show representatives of the four topological types of tilings by convex pentagons and heptagons in which the tiles form two transitivity classes. (Of the 508 types of tiling listed in [3] all but five can be drawn as edge-to-edge tilings with convex tiles, although not all the illustrations in [3] show such tilings; in the notation used there, the five types which can be drawn only using nonconvex tiles are $3_1 3_1 - 2$, $3_2 3_2 - 5$, $3_3 3_3 - 1$, $3_3 3_3 - 2$, and $3_3 3_3 - 3$.) In any finer classification there are probably thousands of types. Even if only edge-to-edge tilings by triangles are considered there are already 38 different types in one finer classification of tilings with two transitivity classes of tiles (see [**11**]). The existence of 503 topologically distinct types of tilings with two transitivity classes of tiles shows that enumeration problems can easily "get out of hand" and become, for practical purposes, unsolvable except possibly with the aid of computers. We believe that there are many thousands of topologically distinct types of tilings by convex tiles which belong to exactly three transitivity classes; clearly, this is more than anyone would wish to work through using only pencil and paper.

In such cases there are still at least two courses open to us. Either we *change the problem* in some way to make the numbers smaller—by insisting that each tile is a regular polygon or placing some other restriction on the tilings under consideration—or better, we *find some general or interesting properties* of tilings which arise naturally from transitivity, or are connected with it, and which can be investigated without the need to carry out a complete enumeration of all possible types.

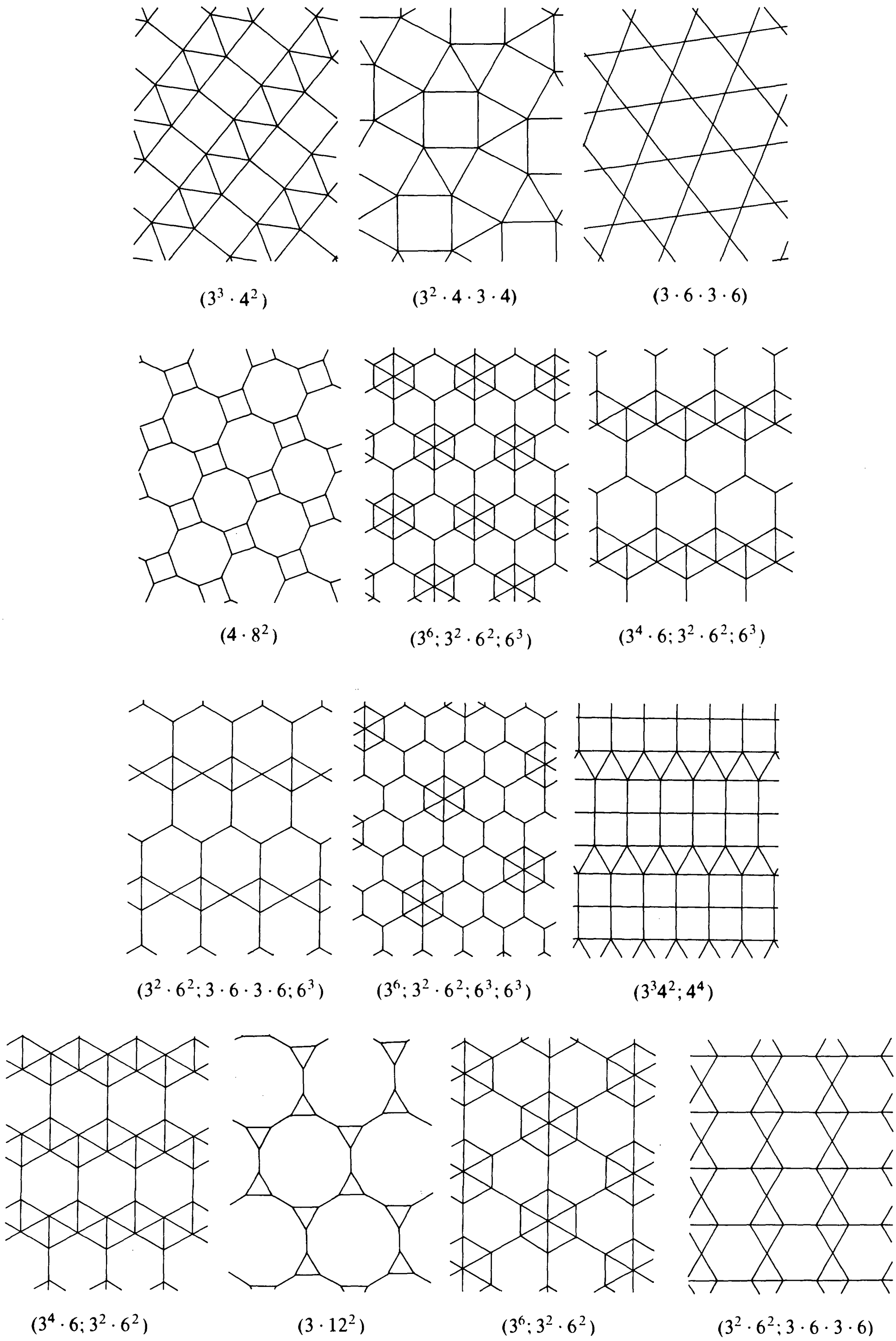


FIGURE 4. **The thirteen types of tilings by regular polygons in which the tiles belong to two transitivity classes.**

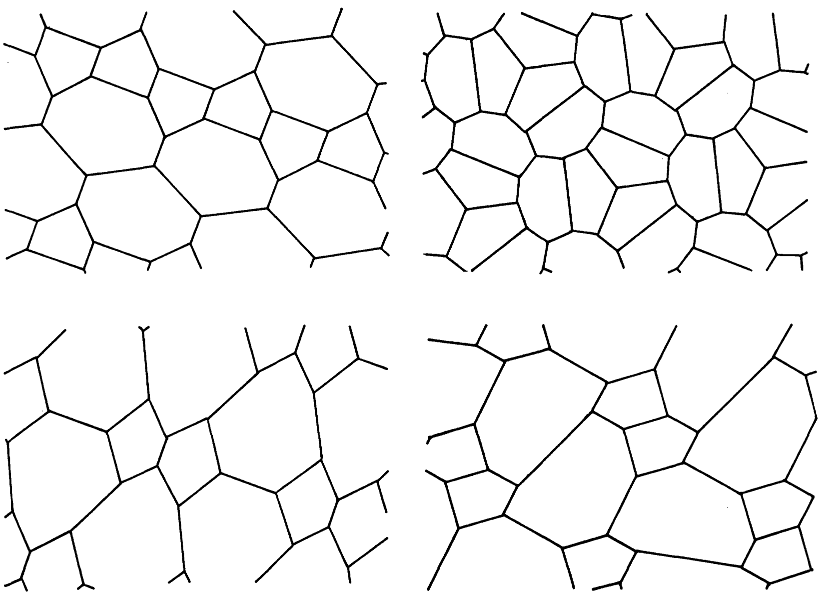
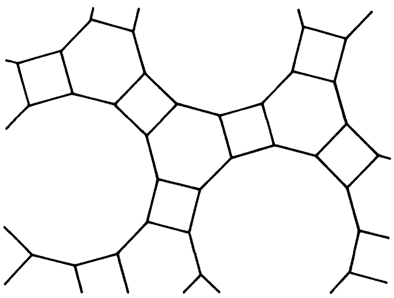
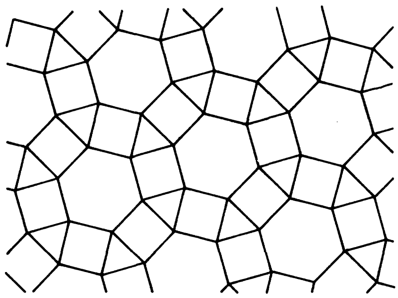


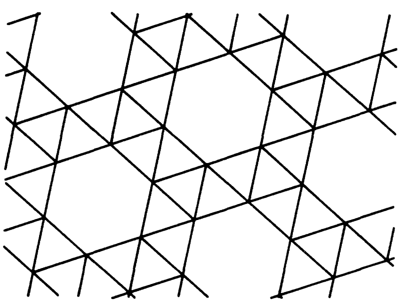
FIGURE 5. **Examples of the four topologically distinct types of tilings by convex pentagons and heptagons in which the tiles form two transitivity classes.**



(a) $(4 \cdot 6 \cdot 12)$



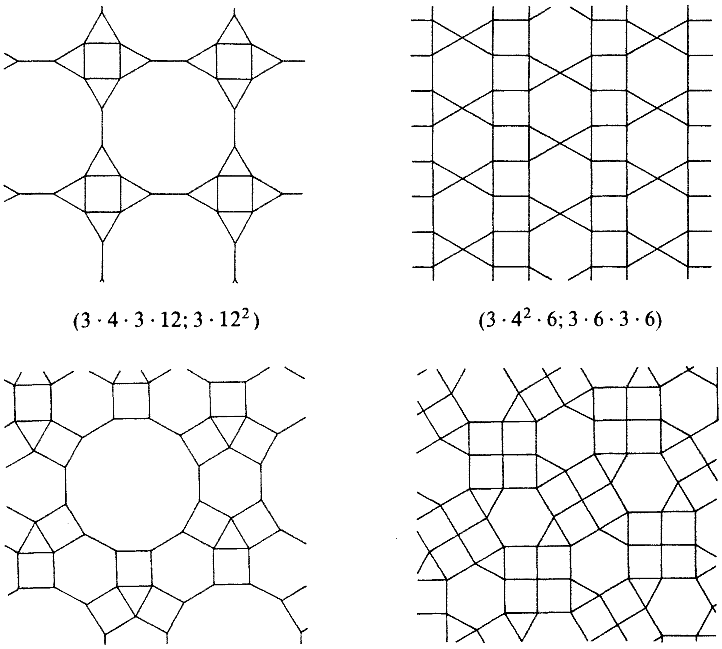
(b) $(3 \cdot 4 \cdot 6 \cdot 4)$



(c) $(3^4 \cdot 6)$

FIGURE 6. **Three examples of tilings by regular polygons, each with three transitivity classes of tiles. In (a) the classes consist of squares, hexagons and 12-gons. In (b) the classes consist of triangles, squares and hexagons. In (c) one class consists of hexagons and two of triangles; the triangles of one transitivity class touch three hexagons, those of the other class touch only two hexagons. Tilings (a) and (b) are equitransitive, but (c) is not because two transitivity classes consist of polygons of the same kind (triangles).**

One promising line of approach is to consider *equitransitive tilings* by convex polygons, that is, tilings whose tiles are convex n-gons for various values of n, and for each n all the n-gonal tiles belong to the same transitivity class. The examples of FIGURE 6 should clarify this concept. The diagram shows three of the "Archimedean" or "uniform" tilings (investigated by J. Kepler in the 17th century) which are conventionally denoted by $(3 \cdot 4 \cdot 6 \cdot 4)$, $(4 \cdot 6 \cdot 12)$, and $(3^4 \cdot 6)$. The first two of these are equitransitive while the third one is not, since in this case the triangles belong to two different transitivity classes. If we insist that the tiles be regular polygons, then it has been shown recently (see [**2**]) that there are exactly 22 types of equitransitive tilings; see FIGURE 7.



$(3 \cdot 4 \cdot 3 \cdot 12; 3 \cdot 12^2)$ $(3 \cdot 4^2 \cdot 6; 3 \cdot 6 \cdot 3 \cdot 6)$

$(3 \cdot 4 \cdot 6 \cdot 4; 4 \cdot 6 \cdot 12)$ $(3 \cdot 4^2 \cdot 6; 3 \cdot 4 \cdot 6 \cdot 4; 4^4)$

FIGURE 7. **There are 22 types of equitransitive tilings by regular polygons, namely, the three regular tilings (FIGURE 2), the thirteen tilings with two transitivity classes of tiles (FIGURE 4), the two equitransitive tilings shown in FIGURES 6(a) and (b), and the four tilings shown here.**

However, in the case of convex tiles without the condition of regularity, it is not hard to see that there are infinitely many types. (In FIGURE 8 we indicate a construction for an infinite family of such tilings.) For all of these the symmetry group is finite and infinitely many different sorts of tiles occur; examples with an infinite symmetry group can also be constructed. Having remarked on this fact, we will probably wish to dismiss such tilings as uninteresting and ask only for equitransitive tilings in which a finite number of different kinds of polygonal tiles occur. It can be shown that any such tiling must be *periodic* (that is, its symmetry group contains translations in at least two different directions). An illustration of a periodic equitransitive tiling is given in FIGURE 9.

Even for periodic equitransitive tilings there are an enormous number (possibly an infinite number?) of different kinds. So our enumerative problem has "blown up" and we must fall back on our second suggestion, namely, that we try to find some general properties of these tilings which are related to equitransitivity. As an example we suggest the following.

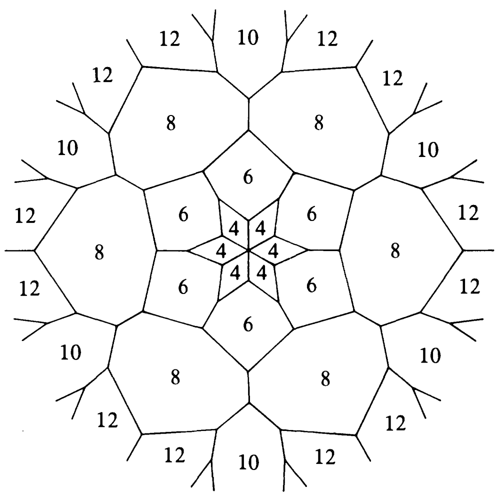


FIGURE 8. **The beginning (center) of an equitransitive tiling with convex polygonal tiles of infinitely many kinds ($2n$-gons for $n = 3,4,5,\ldots$) and a central vertex where m edges meet; here $m = 6$. It is left as an exercise for the reader to show that this construction can be continued over the whole plane in such a way that the tiling is equitransitive.**

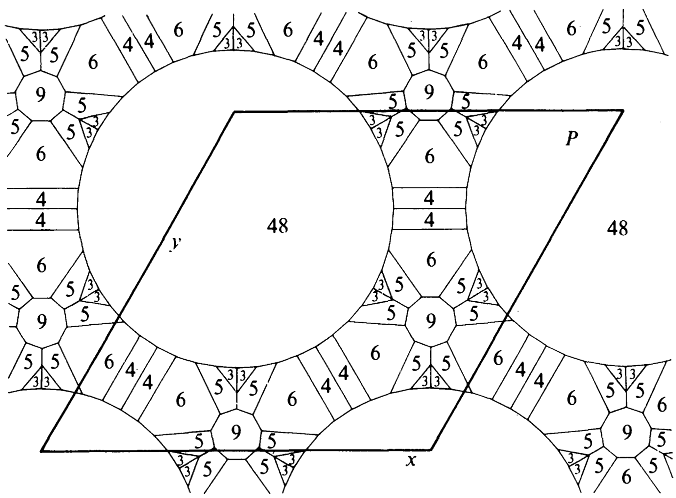


FIGURE 9. **An example of a periodic tiling. A period parallelogram is indicated; the whole tiling can be constructed by parallel displacements $rx + sy$ of this period parallelogram, where x and y are vectors along two adjacent edges of the parallelogram, and r, s are any two integers. The characteristic feature of a periodic tiling is the existence of translations $rx + sy$ in infinitely many non-parallel directions, such that each translation is a symmetry of the tiling. Here the period parallelogram contains $v = 72$ vertices, $e = 111$ edges, and $p = 39$ tiles, fractional parts of edges and tiles being counted as such. In the notation of statement (2) in the text, $p_3 = 12$, $p_4 = 6$, $p_5 = 12$, $p_6 = 6$, $p_9 = 2$, and $p_{48} = 1$.**

CONJECTURE. *In any periodic equitransitive tiling by convex polygonal tiles the maximum number of sides of any tile is* 66.

In other words, a periodic equitransitive tiling may contain 66-gons, but there is no possibility of any k-gonal tiles with $k > 66$. We state this as a conjecture since at the moment we do not know whether or not it is true. It will remain a conjecture until it is proved, in the following sections, when it will become a theorem. Of course, the reader will probably wonder how we could possibly have guessed that 66 is the correct maximum. It may be worth a few lines to explain how we arrived at this number, before attempting to prove the conjecture.

By doodling on scrap paper we found that 48-gons are possible; see FIGURE 9. It is also very easy to prove that no such tiling can contain k-gons with $k > 78$; see Section 2. It became a challenge to narrow the gap between 48 (which is possible) and 79 (which is not). To lower 78 to 72 is not too difficult (see Section 3) but to reduce it to 66 requires rather more delicate arguments; see Section 4. The method of proof suggests a way of finding equitransitive tilings with k-gons, for large k, and following this up led to the discovery of tilings with 66-gons. Hence, we are in the very satisfactory position of not only showing that the conjecture is true, but also that it is, in an obvious sense, the best possible result of this nature. It also has the consequence (details of which we shall not go into) of showing that the number of topologically distinct types of periodic equitransitive tilings is finite, though, of course, the actual number may be extremely large.

To prove the different parts of the conjecture we need to examine equitransitive tilings to see how known results on tilings can be applied to them. In fact we need only two such results—a corollary of Euler's Theorem for Tilings which gives numerical relations between the numbers of n-gons for various values of n, and a knowledge of the periodic (wallpaper) groups of symmetries in the plane. Both will be explained in more detail in the next section. The reader who wishes to learn more about these topics may consult **[8]** or **[10]** for Euler's Theorem, and **[1]**, **[13]**, or **[15]** for the periodic groups of symmetries.

The method of approach to a proof of the conjecture is as follows. We try to break the proof down into a sequence of smaller and simpler problems or steps (often called lemmas) which we then prove in turn. The breakdown can, of course, be accomplished in many different ways and most of these will lead to "dead ends." In published work the abortive attempts at a problem are always suppressed and this leads to the false impression that the authors could see their way through a whole proof at a glance! It has often been said that research is 10% inspiration and 90% perspiration—much of the latter is the work involved in following up false trials which lead only to a large pile of scrap in the wastepaper basket! Of course, with experience one can develop a feeling as to which approaches are likely to lead to positive results, and which are likely to fail. However, except in the simplest problems, nobody is able with certainty to predict the result of any particular approach. The essential features of any investigation are perseverance and stamina.

In keeping with the spirit of this article, from time to time we shall skip formal proofs. It seems much more valuable to explain why we did this or that rather than to fill in every detail and present a highly polished and terse logical argument. Also, of course, the reader may find it interesting, and certainly valuable, to engage in the logical effort of filling in all the missing steps. In any case, the reader may be assured that our assertions are correct even if we do not always present here watertight proofs.

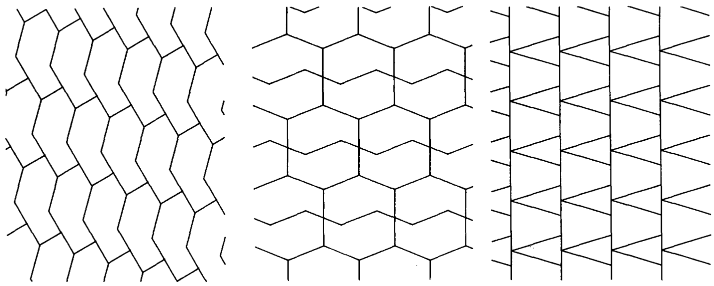
As we proceed we shall mention several directions in which the reader may wish to extend our results—as an intellectual exercise in some cases, or to solve an open problem in others.

2. The first stage in the proof

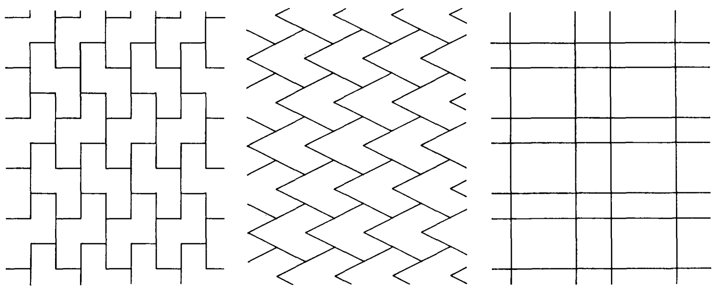
In this and the subsequent sections we shall be concerned only with periodic tilings. We require the following well-known result about symmetry groups of such tilings.

(1) *The symmetry group of any periodic tiling is one of the* 17 *periodic wallpaper groups.*

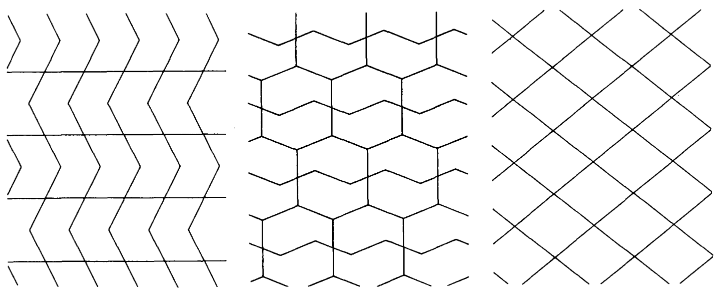
In FIGURE 10 we show examples of tilings whose symmetry groups illustrate the 17 possibilities. It is worthwhile considering these examples carefully and, in each case, identifying the *symmetry elements*: lines of reflection and glide-reflection, centers of rotations of various periods and translations. The letters below the tilings are the international symbols used to denote the symmetry groups: *p*1, *p*2, *pm*, *pg*, *cm*, *pmm*, *pmg*, *pgg*, *cmm*, *p*4, *p*4*m*, *p*4*g*, *p*3, *p*3*m*1, *p*31*m*, *p*6, and *p*6*m*.



*p*1 *p*2 *pm*

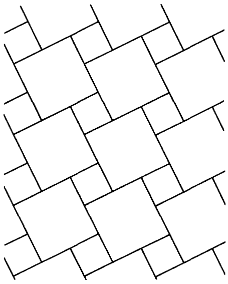


pg *cm* *pmm*

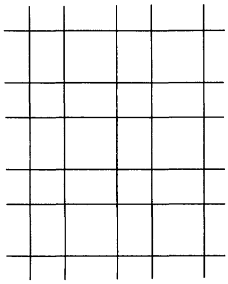


pmg *pgg* *cmm*

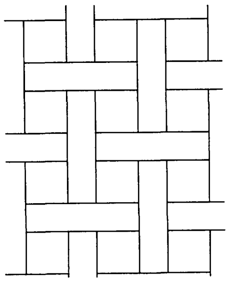
FIGURE 10. **Seventeen tilings whose symmetry groups are the seventeen possible kinds of wallpaper groups.**



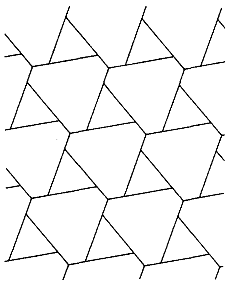
$p4$



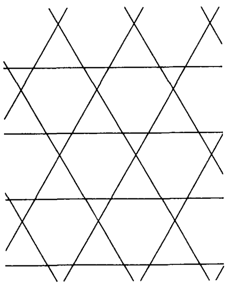
$p4m$



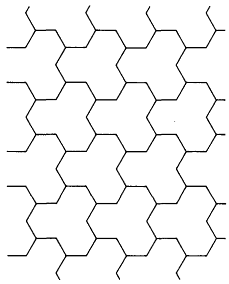
$p4g$



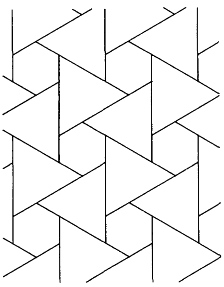
$p3$



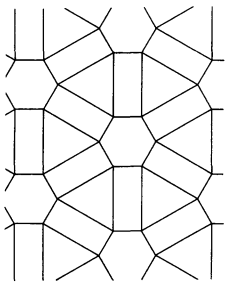
$p3m1$



$p31m$



$p6$



$p6m$

FIGURE 10 **(continued)**

If the two translative symmetries are represented by the vectors x and y, then it is clear that any vector of the form $rx + sy$ (where r and s are integers) will also represent a translative symmetry. If we select x and y in such a way that a parallelogram P, whose sides are x and y, has a minimal positive area (that is, area as small as possible but not zero), then P is called a *period parallelogram* for the tiling. Every periodic tiling has many (in fact, an infinite number of) period parallelograms. In FIGURE 9 we show a tiling (whose symmetry group is $p6m$) on which a period parallelogram is indicated. It will be observed that if we know what the tiling looks like inside a period parallelogram P, then the whole tiling may be reconstructed by applying to P translations of the form $rx + sy$; in this way we cover the whole plane by copies of P and the part of the tiling inside it.

A simplification of the problem is now immediate. Instead of considering the whole tiling we just look at those tiles, or parts of tiles, that lie in a period parallelogram P. For our purposes we need to count the numbers of tiles p, edges e, and vertices v in P, and this is done in the obvious way—if parts of a tile or an edge occur then these are counted as appropriate fractional parts, as in FIGURE 9. The famous Theorem of Euler for Tilings then implies that

$$p - e + v = 0.$$

(We may think of P as bent round onto itself and opposite sides identified—it is then a torus for which the above relation is well known.) Manipulating this equality leads easily to the next statement.

(2) *In any periodic tiling, if the period parallelogram contains p_k k-gons* ($k = 3, 4, 5, \ldots$) *then*

$$3p_3 + 2p_4 + p_5 \geq \sum_{k \geq 7} (k-6) p_k.$$

In fact, equality will hold in this relation if and only if every vertex is an endpoint of exactly three edges. Since p_6 is absent from the inequality, we obtain no information about the number of hexagons. Although the inequality in (2) seems to be well known and proofs of an analogous statement for polyhedra are given in many texts, we were unable to find in the literature any readily accessible proof of the inequality for periodic tilings. For this reason we give details of its derivation in Appendix II.

Statements (1) and (2) apply to any periodic tiling—we must now, in some way, make use of the equitransitivity property. To do this we begin with the following result.

(3) *In any periodic equitransitive tiling, $p_k \leq 12$ for all values of k.*

The proof of this is tedious for we have to consider all the 17 possible symmetry groups, one at a time. For example, let us consider one of the groups, say $p6m$; see FIGURE 11. For any k, take any k-gon in the tiling and let its centroid be represented by C. Under the operations of the

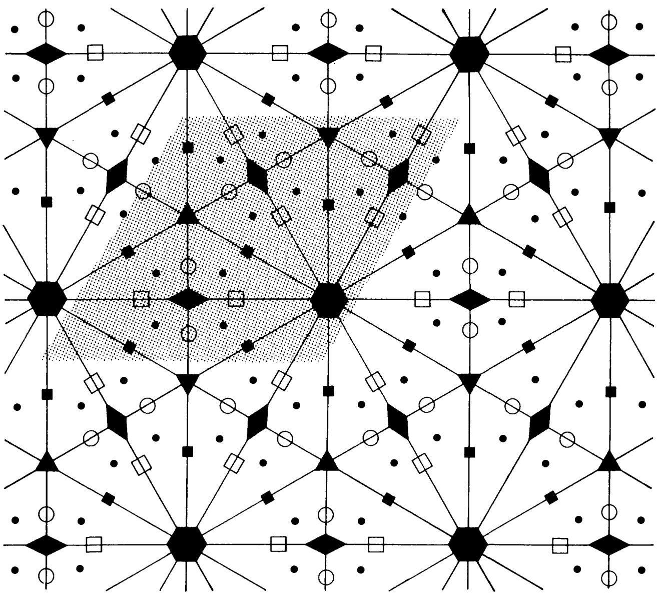


FIGURE 11. The group $p6m$. The lines are lines of reflection. The centers of rotation are marked by small polygons as follows: centers of 6-fold rotation are marked by hexagons, centers of 3-fold rotation are marked by triangles, and centers of 2-fold rotation are marked by rhombs. A period parallelogram P is shown in gray. If a point C is chosen in "general position" (that is, not on any line of reflection) then there are 12 images of C inside P, as shown by dots. If C is chosen on a line of reflection then there are 6 images of C inside P, as indicated by squares, hollow squares and circles. If C is chosen at a center of 6-fold, 3-fold, or 2-fold rotation then there are 1, 2, or 3 images of C in P, as indicated by the hexagons, triangles and rhombs.

group we will get many images of C, and by the equitransitivity property each must be the centroid of a k-gon. The question that concerns us is, how many images of C can occur in the period parallelogram P? To discover this we merely have to start with C in various positions (relative to the symmetry elements) and count! It is not hard to see that the maximum number is 12, and this occurs only with the group $p6m$. Thus statement (3) is true (why?). Notice also that if C is such that only one image lies in P then (for the group $p6m$) C must lie at a center of 6-fold rotation; see FIGURE 11. From this and similar arguments we deduce the next result.

(4) *If the symmetry group is $p6m$ and*

$$\begin{aligned} p_k &= 1 \quad \textit{then } k \textit{ must be a multiple of } 6; \\ p_k &= 2 \quad \textit{then } k \textit{ must be a multiple of } 3; \\ p_k &= 3 \quad \textit{then } k \textit{ must be a multiple of } 2. \end{aligned}$$

(In the last two cases C must lie at a center of 3-fold rotation, or a center of 2-fold rotation, respectively.)

These facts lead immediately to our first objective.

(5) *In any periodic equitransitive tiling, there are no k-gons with $k > 78$.*

Indeed, statement (3) asserts that $p_k \leq 12$ for all k, and combining this with (2) leads to

$$\begin{aligned} (6) \qquad 72 &= 3 \cdot 12 + 2 \cdot 12 + 1 \cdot 12 \\ &\geq 3p_3 + 2p_4 + p_5 \\ &\geq p_7 + 2p_8 + 3p_9 + \cdots + (k-6)p_k + \cdots . \end{aligned}$$

In this last expression, for $k > 78$ the coefficient of p_k is greater than 72. As the left side of the inequality is 72, we deduce $p_k = 0$ for $k > 78$. Therefore, the largest value of k for which p_k is nonzero must satisfy $k \leq 78$, as required.

It is worth noting that if $p_k \geq 2$, then the same inequalities imply $k \leq 42$, so if $k > 42$, then $p_k = 0$ or 1. Moreover, if the group is *not* $p6m$ but one of the other 16 possible symmetry groups, then the left side of (6) is replaced by 48 and so the largest value of k for which $p_k > 0$ satisfies $k \leq 54$. It follows that in our search for the largest value of k for which $p_k > 0$, we need only consider tilings with symmetry group $p6m$ and for which the centroid of the corresponding k-gon lies at a center of 6-fold rotational symmetry. Consequently the only possible values of k in this case are multiples of 6, namely, 78, 72, 66,... .

3. The second stage in the proof

Our next task is to show that, in fact, $p_{78} > 0$ is impossible; hence (4) will imply that $p_k = 0$ for all $k > 72$.

(7) *If a periodic equitransitive tiling with symmetry group $p6m$ contains a k-gon, and $p_k = 1$, then it must contain n-gons for at least $(k-6)/12$ other values of n.*

To establish this we begin by remembering that $p_k = 1$ in the group $p6m$ implies that the centroid of the polygon lies at the center of 6-fold rotation. However, inspection of FIGURE 11 shows that in this case the k-gon can touch at most 6 others of the same kind. (The reader may like to establish this without reference to the diagram.) Moreover, since the k-gon has, as its symmetry group, a dihedral group of order 12, it must touch either 6 or 12 polygons of every other kind. Hence the minimum number of kinds of n-gons (other than the k-gons with which we started) is at least $(k-6)/12$, as required.

Our proof that $p_{78} = 0$ is indirect; we make the assumption that $p_{78} > 0$ and then show that this leads to a contradiction. To carry this out we begin by noting that, according to the remark following (5), $p_{78} > 0$ implies $p_{78} = 1$. Thus, by statement (7), there must be at least $(78-6)/12 = 6$ other kinds of polygons present in the tiling. However, in the proof of (5) we used the fact that $p_n \leq 12$ for $n = 3, 4$, or 5 and $p_n = 0$ for all $n \geq 7$ with $n \neq 78$. Hence the only polygons that can occur in the tiling (other than the 78-gons) are triangles, quadrangles, pentagons, and possibly hexagons. There are thus at most four kinds, which is not enough! We have achieved the required contradiction and so the assumption that $p_{78} > 0$ is incorrect. Hence we have established that $p_{78} = 0$.

Since the next largest possible value of k is 72, we now need to investigate whether or not $p_{72} > 0$ is possible in a periodic equitransitive tiling with symmetry group $p6m$.

4. The third stage in the proof

Let us assume $p_{72} > 0$. We shall show that this leads to a contradiction and so, by reasoning similar to that used in the previous section, we will be able to deduce that $p_{72} = 0$.

We now know that the right side of inequality (6) must terminate; it may be written in the form

$$72 \geq 3p_3 + 2p_4 + p_5 \geq p_7 + 2p_8 + 3p_9 + \cdots + 66p_{72}.$$

If $p_{72} \geq 1$ then this implies that

$$p_7 + 2p_8 + 3p_9 + \cdots + 65p_{71} \leq 6. \tag{8}$$

But we have already remarked that if the centroid C of a polygon does not lie at a center of 2-fold, 3-fold, or 6-fold rotational symmetry in the $p6m$ group, then there will be either 6 or 12 images of C inside P, and the corresponding value of p will be either 6 or 12. Hence,

$$p_k = 0, 6, \text{ or } 12 \qquad \text{if } k = 7, 11, 13, 17, \ldots$$

(that is, whenever k is neither a multiple of 2 nor of 3). The reason for the exclusions in the values of k is that if k is a multiple of 2, then C may lie at the center of 2-fold rotation (and then there will be 3 images of C inside P), and if k is a multiple of 3, then C may lie at a center of 3-fold rotation (and then there will be two images of C inside P). Hence, for the excluded values of k we get:

$p_k = 0$, 3, 6, or 12 if k is a multiple of 2 but not a multiple of 3, that is, if $k = 8$, 10, 14, 16,...

$p_k = 0$, 2, 6, or 12 if k is a multiple of 3 but not a multiple of 2, that is, if $k = 9$, 15, 21, 27,...

$p_k = 0$, 2, 3, 6, or 12 if k is a multiple of 6, that is, if $k = 12$, 18, 24, 30,... .

Trying to fit these numbers into the inequality (8) it is easily seen that if $p_{72} > 0$, then there are only three possibilities, namely, $p_7 = 6$ (and all other p_k are 0), $p_8 = 3$ (and all other p_k are 0), or $p_9 = 2$ (and all other p_k are 0). In each case the tiling contains at most five kinds of polygons other than the postulated 72-gons, namely, triangles, quadrangles, pentagons, hexagons, and either 7-gons, 8-gons, or 9-gons. But from (7) we know that there must be at least $(72 - 6)/12 = 11/2 > 5$ other kinds of polygons. As we have only five other kinds there are not enough; this contradiction establishes that $p_{72} > 0$ is impossible.

5. Completing the proof

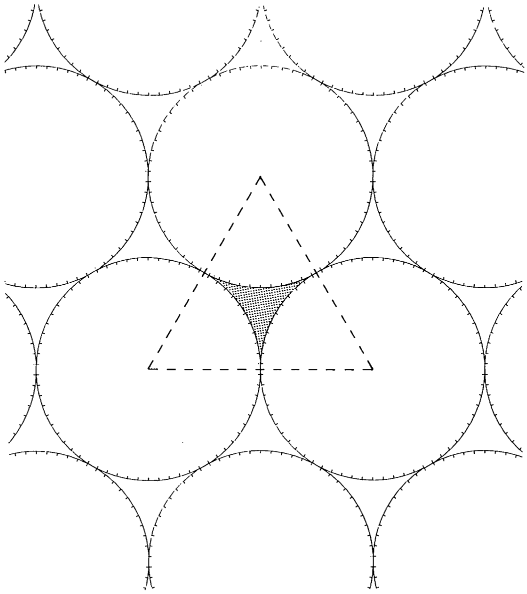
We have shown that $p_k = 0$ for $k > 66$ and the question naturally arises as to whether periodic equitransitive tilings with 66-gons really exist. The method of proof helps us in the search for such tilings; we know that the symmetry group must be $p6m$ and also that each 66-gon must touch six other 66-gons and 12 of each of 5 other types of tiles. With this information, and a little trial and error, we soon arrive at tilings with 66-gons, six of which are shown in FIGURE 12.

We have now established all that we intended and so can state our results as a theorem:

THEOREM. *A periodic equitransitive tiling may contain* 66-*gons, but never contains k-gons for* $k > 66$.

6. Final comments

It will be apparent to the reader who has followed the arguments in the previous sections that we have answered only one of a large number of possible questions relating to periodic equitransitive tilings. Here we mention a number of such problems and state some additional results. In many cases, proofs of the latter involve consideration of a large number of special cases, and there is certainly not room to include details.



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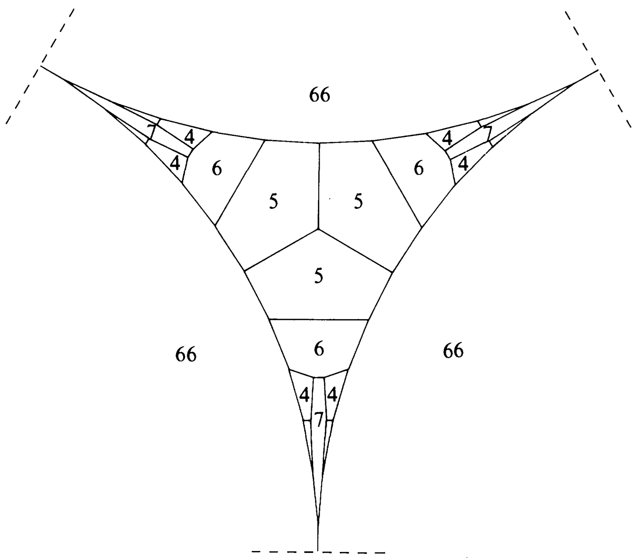
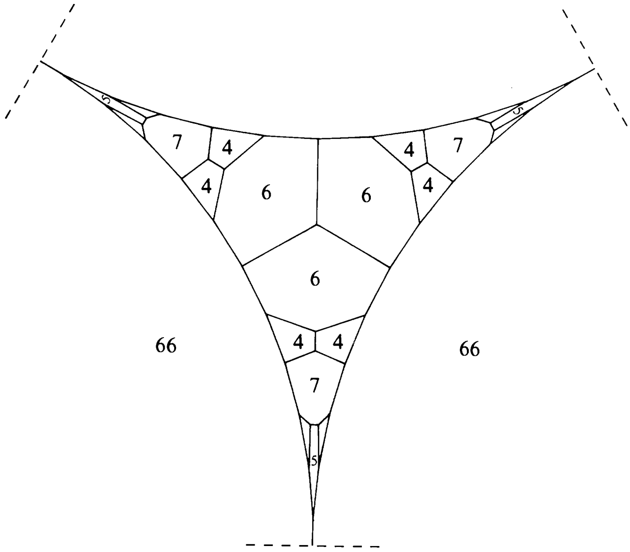


(b)

FIGURE 12. **The six periodic equitransitive tiling, which contain 66-gons. In four of these the 66-gons are arranged as in (a) and then the interstices (one of which is shaded) are tiled as in (b), (c), (d), or (e). In the other two, the 66-gons are arranged as in (f) and the remaining areas (one of which is shaded) are tiled as in (g) or (h). Alternatively, one can construct the whole tiling in each case from the given patch by repeatedly reflecting it in the dashed lines.**



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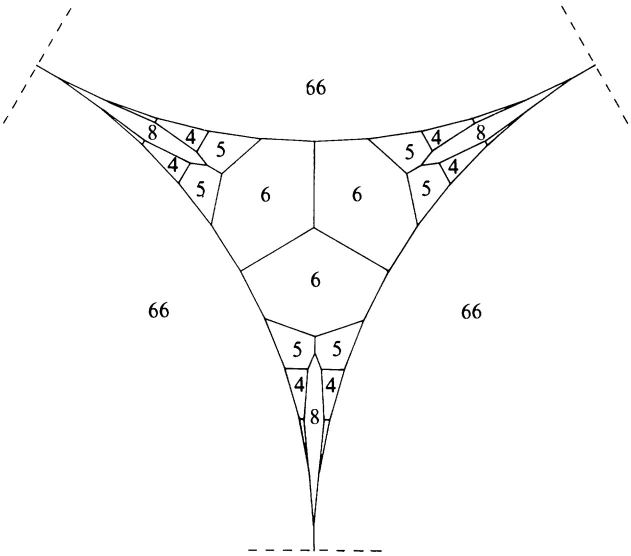
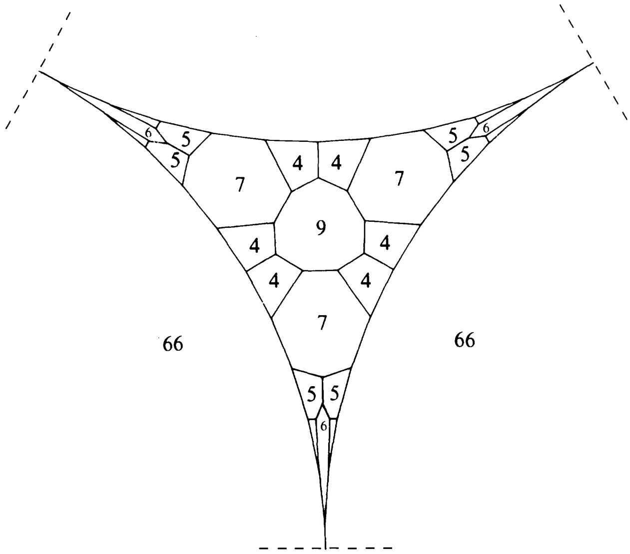


(d)

FIGURE 12 (continued)



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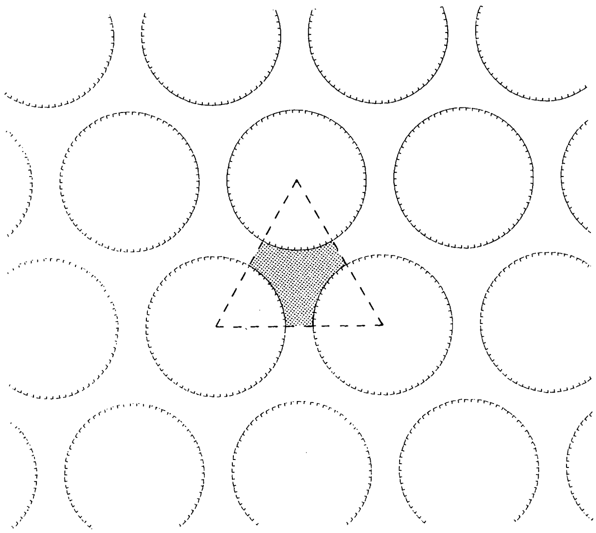
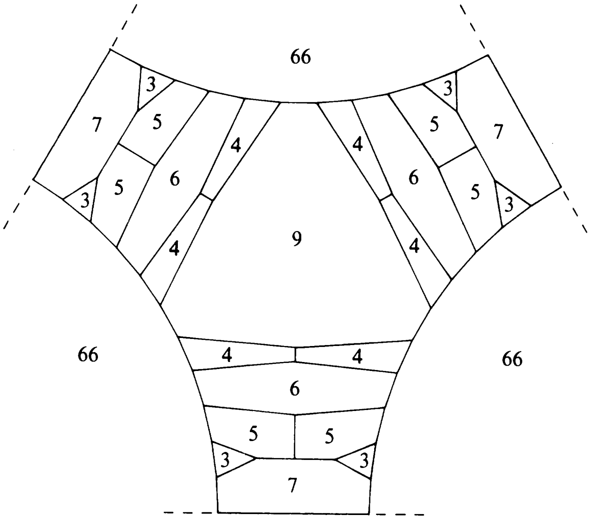
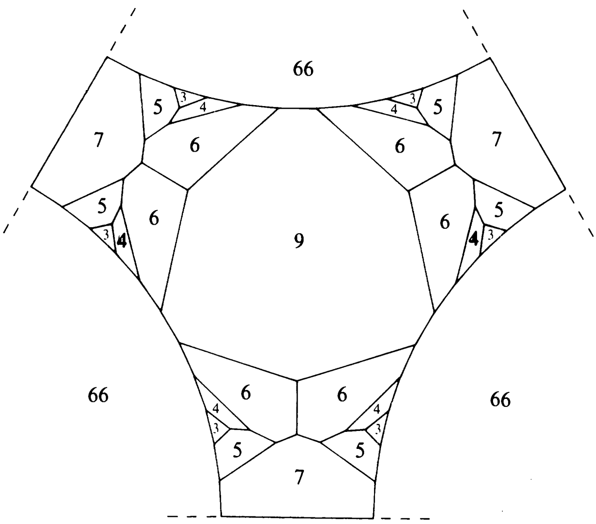


(f)

FIGURE 12 (continued)



(g)



(h)

FIGURE 12 **(continued)**

To begin with, the inequalities $p_k \leq 12$ (see (3)) for all k and $p_k = 0$ for $k > 66$ imply, with a little additional reasoning, that the number of topologically distinct types of periodic equitransitive tilings is finite. On the other hand, we have no idea how many such types exist—we guess it is many millions, perhaps too large a number for all to be listed even with the most sophisticated computers now available. However, this does not preclude the possibility of finding, in some theoretical way, a rough estimate of the number of types—though we must admit that at present we can see no way of doing this.

Other open questions relate to the so-called p-vectors of periodic tilings. These are defined as sequences

$$(p_3, p_4, p_5, p_6, \ldots, p_{66}).$$

Can one characterize, in some way, those p-vectors which correspond to equitransitive tilings? It can be shown that with the additional condition $p_{66} > 0$ there are precisely three such vectors, namely

$$(12,12,6,6,6,0,0,0,0,0,\ldots,0,0,0,1),$$

$$(12,12,12,6,0,6,0,0,0,0,\ldots,0,0,0,1),$$

$$(12,12,12,6,6,0,2,0,0,0,\ldots,0,0,0,1),$$

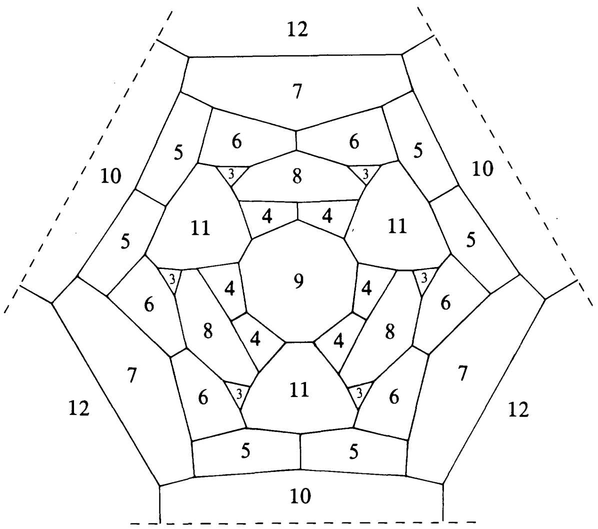
and to these correspond exactly six tilings, namely those shown in FIGURE 12. (The possibility that several tilings correspond to the same p-vector should come as no surprise. But the reader might like to verify that only a finite number of topologically different tilings can correspond to the same p-vector. This fact is essential in establishing the finiteness property mentioned in the previous paragraph.) On the other hand, if we replace the condition $p_{66} > 0$ by $p_{60} > 0$, or $p_{54} > 0$, or $p_{48} > 0$ (or by some other similar condition), then we have no estimate which p-vectors, or how many, can occur. It may be possible, in some cases, to answer this question by careful examination of the many cases that can arise. But this is tedious and probably will not lead to any general characterizations.

Instead of looking for periodic equitransitive tilings with k-gons for the largest possible value of k, one could ask for such tilings which contain k-gons for the largest possible number of *different values* of k. Here the answer is known to be ten; there are precisely two topologically distinct tilings which include k-gons for the ten values $k = 3, 4, 5, 6, 7, 8, 9, 10, 11$ and 12. These are shown in FIGURE 13.

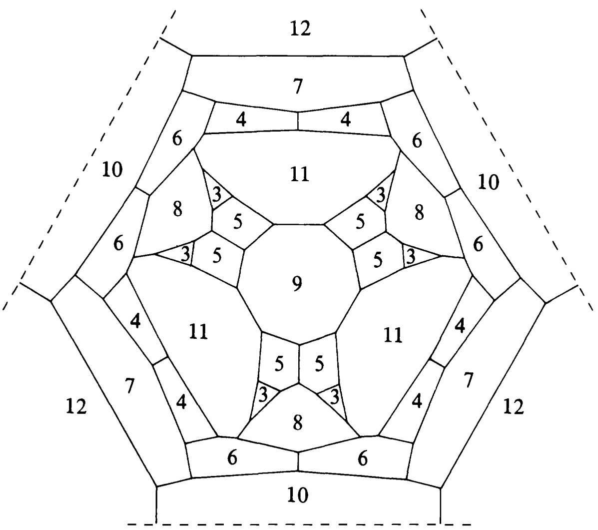
The fact that $p_k = 0$ if $k > 48$ and k is not a multiple of 6 suggests the problem of determining for which values of k do k-gons occur in *some* periodic equitransitive tiling. In fact one can show, again by considering many special cases, that there are just 34 such values of k, namely,

$$3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,20,21,22,24,26,27,28,$$
$$30,32,33,36,40,42,44,48,54,60,66.$$

Many more problems arise if we agree to generalize the kinds of tilings under consideration. For example, we may dispense with the requirement that the tiles are convex or polygonal. Then the conditions usually imposed are that each tile is a closed topological disk, each tile has diameter less than a given constant and incircle radius greater than some other constant. If, moreover, any two tiles either do not meet, or meet in a single point ("vertex of the tiling") or a single arc ("edge of the tiling"), then the tiling is called *normal* and a tile is said to have k *adjacents* if its boundary consists of k edges of the tiling. We can now consider the analogues of the questions discussed above for normal periodic equitransitive tilings, replacing "k-gons" by "tiles with k adjacents." In view of certain results (see **[12]**, **[16]**) on the existence of tilings by convex tiles of the same topological type as any given normal periodic tiling, it appears reasonable to conjecture that our theorem remains valid for such tilings—but the details of the proof have not been worked out. Probably all the other facts mentioned above remain valid—with reasonable changes in wording—for such tilings.



(a)



(b)

FIGURE 13. **Two periodic equitransitive tilings using ten different kinds of convex polygons as tiles. The 12-gons are arranged in a similar manner to the 66-gons in FIGURE 12(f) and the remaining part of the plane is tiled by copies of the patch shown here.**

On the other hand, if we generalize still further and allow in our tilings pairs of tiles which meet in more than a single vertex or edge of the tiling, and also admit the possibility for a tile to be a "digon" (have just two adjacents), then tilings such as the one shown in FIGURE 14 need to be investigated. In this situation, of course, our theorem is no longer valid and the tiling may contain tiles with more than 66 adjacents. It may be of some interest to the reader to discover exactly where the proof of the theorem breaks down. The tiling in FIGURE 14 contains tiles with 120 adjacents, but we do not know whether this is the largest possible number.

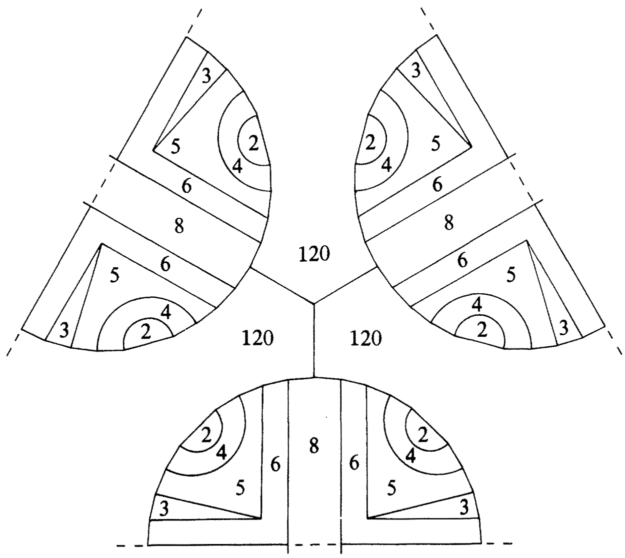


FIGURE 14. **If we relax the condition that all the tiles are convex, and allow multiple contacts as shown here, it is possible for tiles with $k > 66$ adjacents to occur in periodic equitransitive tiling. In this example some tiles have 120 adjacents. As in the previous two diagrams, the whole tiling can be constructed by repeatedly reflecting the given patch of tiles in the dashed lines.**

Other interesting problems arise by specialization. Since the proof of our theorem showed that 66-gons are possible only if the symmetry group of the tiling is $p6m$, a natural question concerns the maximal possible number of sides in tiles of a periodic equitransitive tiling with symmetry group $p4m$, or with any other given group. We conjecture that for $p4m$ the maximal number is 48 and that for $p1$ it is 12—but we have not established this, or the bound for any group other than $p6m$.

We hope that these remarks have given the reader an indication of our answer to the question posed at the beginning of this note: "How does one do research in mathematics?" The general approach we have suggested applies to most branches of mathematics: research usually begins with an empirical investigation which enables one to guess that certain statements may be true. The second step is to establish the truth of these conjectures by sound logical reasoning. Frequently the conjectures have to be modified in carrying out this latter stage—perhaps extra conditions have to be imposed to eliminate unwanted cases that were overlooked in the initial investigation, or perhaps the results turn out to be more general than was at first supposed. However, the only truly satisfactory way to learn about research is to try it for oneself; this is frequently an enjoyable and rewarding experience.

The authors are indebted to the referees for many suggestions leading to improvements in the presentation.

Appendix I

The **Euclidean plane** is the familiar plane of elementary geometry; from our point of view the essential features are that length, angles and parallels are defined. A **tiling** is any family of sets $\{T_1, T_2, \dots\}$ which covers the plane without gaps or overlaps. More precisely this means that every point of the plane belongs to at least one **tile** T_j, and any point which belongs to more than one tile necessarily lies on the boundary of each.

A polygon is **convex** if each of its interior angles is strictly less that 180° (or π radians). It is called **regular** if all its sides are equal in length and all its angles are equal. In this paper we are mainly concerned with tilings in which each tile T_j is a convex polygon. Such a tiling is called **edge-to-edge** if whenever two tiles meet they do so in a side or a corner of each.

Any rigid motion of the plane which maps the tiling onto itself is called a **symmetry** of the tiling. By "rigid motion" we mean one that preserves distances. Such motions are of one of four kinds: translations, reflections, glides, or rotations. For example, in FIGURE 1 there is a rotation of half a turn which maps tile A onto tile B and, clearly, this will map every tile of the tiling onto some tile. This rotation is, therefore, a symmetry of the tiling. It is well known that motions can be compounded, and this imparts an algebraic structure to the set of symmetries of a tiling—this structure is known as a **group**. The group of all symmetries of a tiling is known as its **symmetry group**.

To follow the discussion in the text it is essential that one should be able to recognize without difficulty the symmetries of a given tiling and also become familiar with the concept of a transitivity class. For this, the reader to whom these ideas are new is urged to study FIGURES 1 to 4 and 6 to 9, and to make sure he understands the assertions about the transitivity classes that are made in the captions.

The final notion to be explained is that of the **topological type** of a tiling. We say that two tilings are of the same topological type if it is possible to set up a correspondence (in which each tile of the first tiling corresponds to one in the second tiling and conversely) with the following property: whenever the intersection of a set of tiles in one tiling is a vertex or an edge, then the same is true of the corresponding set of tiles in the other tiling. For example, in FIGURE 3 the tilings (e), (f), (g) and (h) are of the same topological type—it will be noticed that in each six tiles meet at each vertex, and using this it is easy to set up the correspondence between the tiles required in the definition. On the other hand, none of the other four tilings in FIGURE 3 is of the same topological type as these four. In fact, the eight tilings shown belong to precisely four topological types. Any two tilings which are not of the same topological type are said to be **topologically distinct**.

Appendix II

To prove (2) we consider the period parallelogram P. Since each k-gon has k sides and each edge of the tiling is a side of exactly two polygons, we obtain $2e = \Sigma k p_k$. (When no limits are specified, the summation is over all values of k greater than or equal to 3.) In a similar manner, since each vertex is a corner of *at least* three polygons and each k-gon has k corners, we arrive at the inequality $3v \le \Sigma k p_k$. Substituting these values in $6p - 6e + 6v = 0$ and writing $p = \Sigma p_k$ yields

$$6\sum p_k - 3\sum k p_k + 2\sum k p_k \ge 0,$$

which is, after slight rearrangement, the inequality given in (2).

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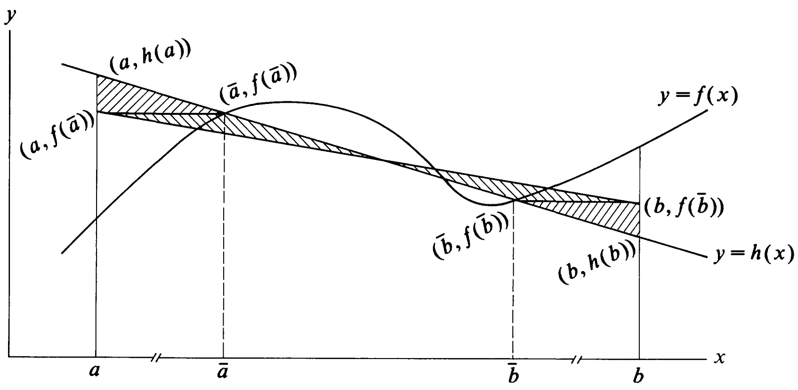
Proof without words:
The Gaussian Quadrature as the Area of Either Trapezoid

$$(1/2)(b-a)(f(\bar{a})+f(\bar{b}))=(1/2)(b-a)(h(a)+h(b))$$



—MIKE AKERMAN
University of Portland

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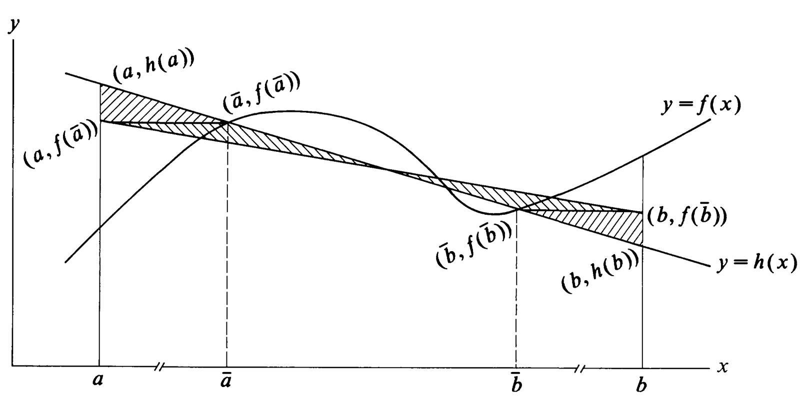


—MIKE AKERMAN
University of Portland

Mu Torere: An Analysis of a Maori Game

Marcia Ascher
Ithaca College
Ithaca, New York 14850

Mu torere is a game played by the Maori, the indigenous people of the region now called New Zealand. The game predates the late eighteenth-century arrival of Europeans. *Mu torere* is usually translated as draughts (the British name for the game Americans call checkers) with the qualification that the two games are considerably different [1], [3], [4], [5], [8], [9], [10], [11]. The game attracts our interest from several points of view. As with any game, we can simply play it for enjoyment. To be enjoyable, it must have sufficient, but not too much, complexity. Some games, such as tic-tac-toe, soon lose enjoyment because a winning or tying strategy is easily learned. Other games, such as chess, require too much skill and concentration to be broadly appreciated. What is involved in the game and its level of complexity can be seen more clearly by analysis. Analysis also raises combinatorial questions that are interesting for themselves. When examining a game that has been a long time favorite, we surmise that there is something in it that has captured and held attention. Anthropologists can state the context of the game, the rules and the observed attitudes of the players; using mathematics we can show the implications of the rules and so gain understanding of what the players find engaging. Our additional insights will be of anthropological interest because games of strategy are a rarity among Oceanic peoples. Further, our analysis resolves an inconsistency in the literature regarding the starting rules.

Mu torere is played on an 8-pointed star design. There are two players and each has four markers. The etymology of the name of the game remains obscure and its play is unceremonious. The star design is easily and readily drawn on sand, wood, or bark with anything that comes to hand such as a piece of charcoal or the point of a nail. The markers can be pebbles or, since post-contact times, bits of broken china or even pieces of potato as long as they are distinguishable by the players [3], [4], [8], [9]. The center of the design (see Figure 1) is generally referred to

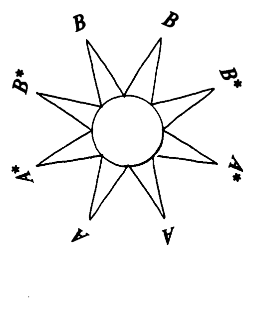


Figure 1. ***Mu torere*** **playing design with players' markers in starting positions. (The first two moves of each player must be made by one of the markers with an *.)**

as *putahi* and the emanating rays as *kawai*. As with all words, their meaning depends on their context. Other than denoting the parts of this game design, *putahi* means to join or meet as paths or streams running into each other and *kawai* is variously translated as shoots or branches of a creeping plant, tentaculea of a cuttlefish, or handles of a basket [**10**], [**12**]. Here we call the players A and B and identify their markers by these letters. To begin, the markers are placed as shown in FIGURE 1. Players move, one marker per move, alternately. A move can be made to an empty adjacent star point or to the center if it is empty. The game is won when the opponent is blocked so that he cannot move. The blocked player is *piro* which, when associated with a game, translates as 'out' or 'defeated.' (Otherwise, *piro* translates as 'putrid' and 'foul smelling' [**10**], [**12**].) On the first two moves of each player, only their outer markers (those with an * on the figure) can be moved so that no player is quickly blocked [**1**]. Some writers [**3**], [**8**] report that only the first move of each player is so restricted but that cannot be the case since the game could then always be won on the second move of the second player.

We approach the analysis by first simplifying the game to a 2-point star design with only one marker for each player. Calling the number of markers per player n, the number of star points is $2n$, and we have simplified the game to the case $n=1$. After this analysis, we will consider, in turn, the cases $n=2,3$, and beyond. We define a **board configuration** by a sequence of $2n+2$ letters. The first letter designates the player with the next move (A or B); the second is the occupant of the center (A, B, or O for no one); and the next $2n$ letters are the markers filling the star points (A, B, or O) read consecutively. For purposes of analyzing the game, configurations are considered equivalent regardless of where reading begins on the star and whether the points are read clockwise or counterclockwise. This takes into account the symmetry of the star and marker placement; any actual game configurations which are related by rotation or reflection are considered equivalent.

$n=1$

For $n=1$, there are six possible configurations: ① $AOAB$; ② $BAOB$; ③ $AAOB$; ④ $BOAB$; ⑤ $ABOA$; and ⑥ $BBOA$. A game must start with configuration ① or ④ and can

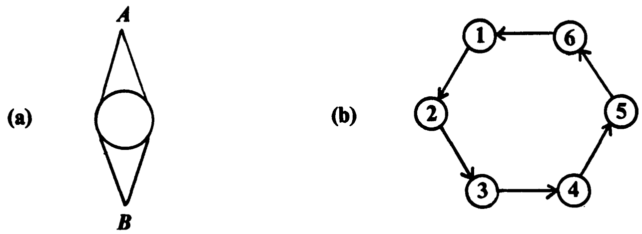


FIGURE 2. (a) Initial marker positions for the game with $n=1$. (b) The sequence of board configurations for the game; no winning configurations exist.

follow no other pattern than that shown in FIGURE 2. All possible configurations must be encountered, there are no choices in the game, and there are no winning configurations. We conclude that $n=1$ would not be an enjoyable game.

$n=2$

For $n=2$, there are 12 possible configurations: ① $AOAABB$; ② $BAOABB$; ③ $AAOBAB$; ④ $BOABAB$; ⑤ $ABOABA$; ⑥ $BBOAAB$; ⑦ $ABOAAB$; ⑧ $BBOABA$; ⑨ $AOABAB$; ⑩ $BAOBAB$; ⑪ $AAOABB$; and ⑫ $BOAABB$. The game begins with configurations ① or ⑫.

All possible configurations may be encountered but, since there are two branch points where choices are made by the players, not all must be encountered. (See FIGURE 3.) Again, there are no winning configurations and so it would not be an enjoyable game. We can observe a point of symmetry at the center of the flow figure. Either A or B moves first but each is followed by the same pattern of moves. Relative to the two possible starting points, each configuration can be paired with its **complement**, where the complement of a configuration is obtained by interchanging A's and B's. Thus, ① and ⑫, ② and ⑦, etc. are complements.

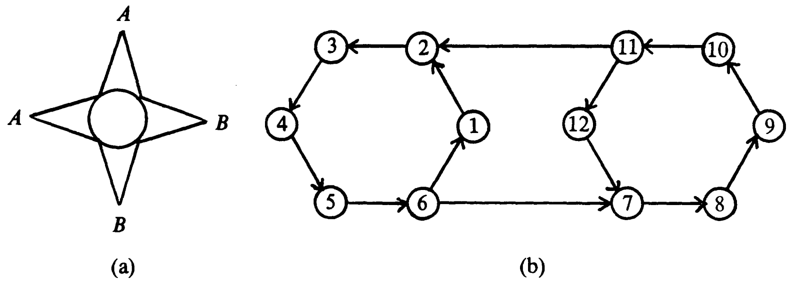
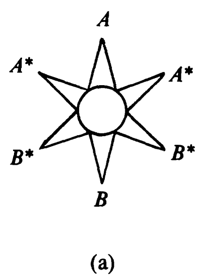


FIGURE 3. (a) Initial marker positions for the game with $n = 2$. (b) The sequence of board configurations for the game; no winning configurations exist.

$n = 3$

The game for $n = 3$ could be enjoyable. FIGURE 4 contains a list of all possible configurations and FIGURE 5 shows the flow of the game. The game must start at ① or ㉕ and cannot immediately terminate in a win since the rule about initial moves bars configuration ③ or ㉖ as the next step. The shortest game could be won via the route ① → ② → ④ → ⑤ → ⑥ → ⑦ → ① → ③ but that would involve a poor choice on B's part in going from ⑤ to ⑥. At that juncture, the alternative choice for B is ⑲ which soon involves A and B in



① *AOAAABBB*	⑪ *AAOBABBA*	㉑ *AOBABABA*
② *BAOBBBAA*	⑫ *BAOBBABA*	㉒ *BAOBABAB*
③ *BAOABBBA*	⑬ *AAOBABAB*	㉓ *BAOBAABB*
④ *AAOBBAAB*	⑭ *BOABABAB*	㉔ *AAOBBBAA*
⑤ *BOBBAABA*	⑮ *ABOABABA*	㉕ *BOBBBAAA*
⑥ *ABOABBAA*	⑯ *BBOAABAB*	㉖ *ABOBAAAB*
⑦ *BBOAAABB*	⑰ *ABOBAABA*	㉗ *ABOAAABB*
⑧ *BBOAABBA*	⑱ *BBOABAAB*	㉘ *AAOBBABA*
⑨ *AOAABBAB*	⑲ *ABOAABAB*	㉙ *AAOABBBA*
⑩ *BAOABBAB*	⑳ *BBOABABA*	㉚ *BBOBAAAB*

(b)

FIGURE 4. **(a) Initial marker positions for the game with $n = 3$. (The first two moves of each player must be made by one of the markers with an *.) (b) The thirty combinatorially possible configurations. If A begins, this is configuration ① and if B begins, it is configuration ㉕. Note that every configuration has a complement obtained by interchanging A's and B's; thus ① and ㉕ are complements.**

additional choices. Looking at FIGURE 5, we can see that there is only one winning configuration for each player and that different lengths of play can lead to either of them or the game can go on indefinitely. The diagram has a central point of symmetry and, together with the list of configurations, can be examined for a variety of configuration pairings. Also observe that although (29) and (30) are possible configurations, they do not appear on the flow-figure as no permissible moves lead to them.

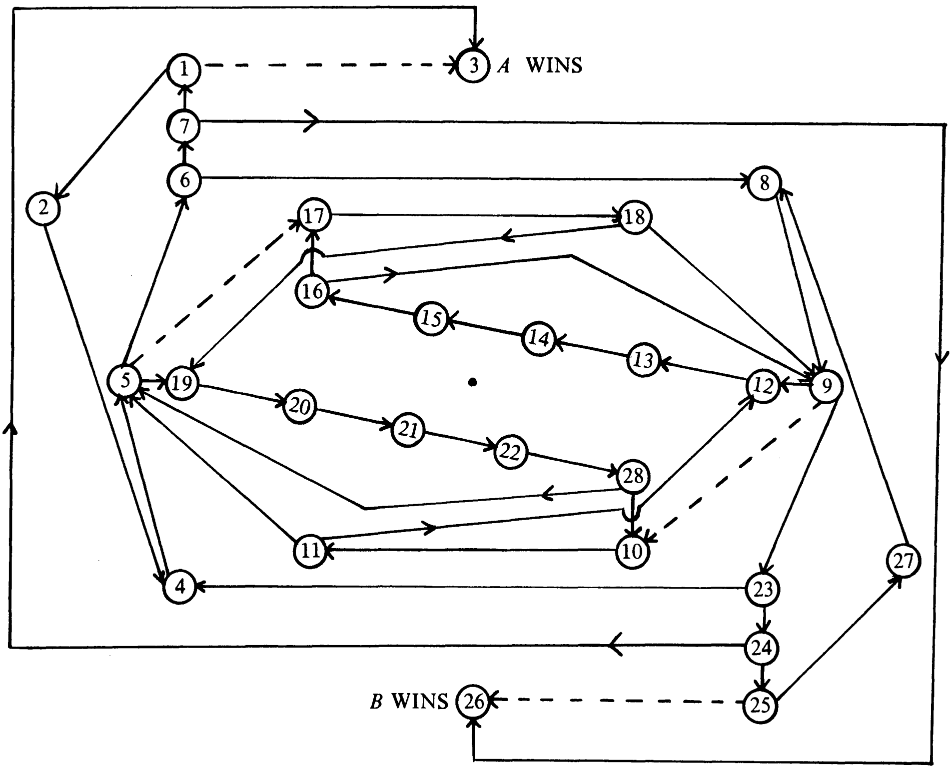


FIGURE 5. **Play for $n = 3$ starts at (1) if A begins or at (25) if B begins. (A dotted path is not permitted as a player's first or second move.) Configurations which are symmetric with respect to the center point of the diagram are complements (e.g., the pairs (10) and (17), (4) and (8) are complements).**

Analysis of the game for $n > 3$

Instead of listing and drawing configurations for the game for higher values of n, we turn to questions that will help analyze the game for any n.

(i) How many winning configurations are there for each player?
(ii) How many configurations are possible but never attained in play?
(iii) How many total configurations are possible?
(iv) Assuming best play by both players, is there a forced win for one of them?

Since configurations are considered equivalent regardless of where on the star the reading begins, we will standardize our readings always to begin at the empty position on the star if one exists. Thus, we need not be concerned with equivalence due to rotation unless all the star positions are filled and the center is empty.

(i) *Winning configurations.*

In order for A to win, B must be blocked from moving. This can only occur when it is B's turn to move, the center is filled by A, and the empty position on the star has A on both sides of it. Therefore, a winning configuration for A must be $BAOA\dots A$ with $2n-3$ positions between the A's. The $2n-3$ positions contain $n-3$ A's and n B's. There are $(2n-3)!/(n-3)!n!$ different possible orderings for $n\geqslant 3$ and no possibilities for $n=1,2$. As was noted before, different orderings that are due to reading star points clockwise and counterclockwise are equivalent and so reversals must be eliminated from the count. This can be accomplished by division by 2 except for orderings that are their own reversals. When n is odd, a self-reversal must have a B in its center position and the same $n-2$ elements in reversed order on either side. On the other hand, when n is even, $n-3$ is odd so an A must be central. Calling the number of self-reversals Q and the number of A's winning configurations $W(A)$, we see that

$$W(A)=\begin{cases}\dfrac{1}{2}\left(\dfrac{(2n-3)!}{(n-3)!n!}+Q\right) & \text{for } n\geqslant 3,\\ 0 & \text{for } n=1,2,\end{cases}$$

$$Q=\begin{cases}\dfrac{(n-2)!}{\left(\frac{n-3}{2}\right)!\left(\frac{n-1}{2}\right)!} & \text{for } n \text{ odd},\\[2ex] \dfrac{(n-2)!}{\left(\frac{n-4}{2}\right)!\left(\frac{n}{2}\right)!} & \text{for } n \text{ even}.\end{cases}$$

Writing this result more compactly we get:

$$W(A)=\begin{cases}\dfrac{1}{2}\left(\dbinom{2n-3}{n}+\dbinom{n-2}{[n/2]}\right) & \text{for } n\geqslant 3,\\ 0 & \text{for } n=1,2.\end{cases}\tag{1}$$

The complements of the winning configurations for A are the winning configurations for B so $W(B)=W(A)$.

(ii) *Unattainable configurations.*

A configuration in which B is blocked cannot result from a move by B. Any move that B made to get into the configuration could simply be retraced. Therefore, configurations that are the same as the winning configurations of A but differ by having the next move to be made by A, can never be attained. That is, $AAOA\dots A$ are unattainable configurations. By similar reasoning, any unattainable configurations with A in the first position must be of this form. Thus, $U(A)$, the number of unattainable configurations with A in the first position, must equal $W(A)$ and, by complementarity, $U(B)=W(B)$.

(iii) *Total number of configurations.*

The "center-filled" configurations and the "center-empty" configurations require different analyses. For the first, our standardized reading beginning at the empty position on the star avoids the problem of replication by rotation which we encounter in the second.

a. Center filled.

Assume that A has the next move and that A is in the center. The configurations have the form $AAO\dots$ where the remaining $2n-1$ positions contain n B's and $n-1$ A's. Using reasoning similar to that used to count the winning configurations, the orderings are counted and we divide by 2 to account for duplication by reversal. Denoting the resulting number of configurations by $N(AA)$, we have

$$N(AA)=\frac{1}{2}\left(\binom{2n-1}{n}+\binom{n-1}{[n/2]}\right) \qquad n\geqslant 1. \tag{2}$$

The same number of configurations would result for B in the center with A having the next move, and for each of A or B in the center with B having the next move. Thus, the total number of configurations with the center filled is $4N(AA)$.

b. Center empty.

Assuming that A has the next move, we see that the configurations with empty centers are $AO\ldots$ where the remaining $2n$ positions contain n A's and n B's. Our problem now is to count the number of distinct orderings of these repeated items arranged in a circle where orderings are equivalent if they are the same after rotation and also equivalent if they are the same when read clockwise or counterclockwise.

The $2n$ markers can be arranged in $\binom{2n}{n}$ different ways. Many of them, however, are rotations of each other. For example, for $n=2$, $AABB$, $ABBA$, $BBAA$, and $BAAB$ are the same arrangement rotated $0,1,2,3$ times. Since each arrangement usually becomes another each time it is rotated, division by $2n$ would seem to get rid of the multiplicity and give the number of distinctly different arrangements. However, before dividing by $2n$, we must account for the arrangements that are invariant under rotation (such as $ABAB$, $BABA$, $ABAB$, $BABA$ for $n=2$). This repetition occurs when a circular arrangement is made up of a repeated string of A's and B's whose length is a divisor of $2n$. The number of repetitions of the string must also be an exact divisor of n. The number, $N(R)$, of distinctly different arrangements, inequivalent under rotation is:

$$N(R)=\frac{1}{2n}\sum_{d\mid n}\phi(d)\binom{2n/d}{n/d},$$

where $\phi(d)$ is the totient function; that is, $\phi(d)$ is the number of positive integers less than d and relatively prime to d, and $\phi(1)=1$. A detailed discussion of this result as an application of Pólya's theorem can be found in [**2**, pp. 199–208].

To account for equivalence of arrangements under "reversal" (i.e., the same arrangement may appear different when read clockwise and then counterclockwise), $N(R)$ must be modified. This will proceed differently for odd and even n.

Before we divided them into $2n$ equivalent classes, there were $2nN(R)$ arrangements due to an original set of all possible arrangements and those added because of self-replication. Now, with reversals of the original set and of each of the rotated sets, there will be $4n$ equivalent classes rather than $2n$. For example, for $n=4$, there are 16 circular arrangements equivalent to $AAABBABB$, namely, itself, its 7 rotated forms, its reversal $BBABBAAA$, and the reversals of the 7 rotations. But again, before dividing by $4n$ we must account for self-replication. Within the set of all possible arrangements, those that are self-reversals have n elements, $n/2$ A's and $n/2$ B's, symmetrically placed on each side of the center. This can occur in $\binom{n}{n/2}$ ways and only for n even. Each time the original set of all possible arrangements is rotated by one position, it again yields the set of all possible arrangements. Within each rotated set, therefore, there are also $\binom{n}{n/2}$ self-reversals. For n even, then, the number of self-reversals is $2n\binom{n}{n/2}$ and the number of distinct arrangements where rotations and reversals are equivalent is N_e where

$$N_e=\frac{2nN(R)+2n\binom{n}{n/2}}{4n}=\frac{1}{2}\left(N(R)+\binom{n}{n/2}\right). \tag{3}$$

For n odd, there can be no arrangements that are their own reversals. Instead we find that the reversals of some arrangements duplicate a rotation of the same arrangement.

Let X be an arrangement, XC^i its rotation clockwise i places, where $i \leqslant n-1$, and $V(XC^i)$ its reversal. Then, in terms of the elements in the arrangement,

$$X = x_1 x_2 \cdots x_{2n},$$
$$XC^i = x_{i+1} x_{i+2} \cdots x_{2n} x_1 \cdots x_i,$$
$$V(XC^i) = x_i \cdots x_1 x_{2n} \cdots x_{i+2} x_{i+1}.$$

In order for an XC^i to be the reversal of XC^{i+1}, the following elements would have to be equal:

$$x_{1+k} = x_{2i+1-k}, \qquad k = 0,\dots,i-1 \qquad \text{for } i = 0,\dots,n-1$$
$$x_{2n-k} = x_{2i+2+k}, \qquad k = 0,\dots,n-i-2 \qquad \text{for } i = 0,\dots,n-2.$$

For each i there are $n-1$ equations to be satisfied. Since n is odd, there are available $(n-1)/2$ pairs of A's and $(n-1)/2$ pairs of B's and so these $n-1$ equations can be satisfied in $\binom{n-1}{(n-1)/2}$ different ways. The same equations guarantee that

$$XC^{i-m(\bmod 2n)} = V(XC^{i+1+m}) \qquad m = 1,\dots,(n-1)/2.$$

Therefore, when an arrangement is the reversal of its rotation clockwise by one position, each of its rotations is the reversal of another rotation of it. An arrangement and its reversal differ by rotation through an odd number of positions. For example, for $n = 3$, the 6 arrangements $BBBAAA$, $BBAAAB$, $BAAABB$, $AAABBB$, $AABBBA$, and $ABBBAA$ are equivalent under rotation. Since $BBAAAB$ is the reversal of $BAAABB$, it follows that $BBBAAA$ and $AAABBB$ are reversals and $ABBBAA$ and $AABBBA$ are reversals. A similar examination of their elements shows that no arrangement can be the reversal of its rotation through an even number of positions. Therefore, for n odd, the number of rotation classes within which there are self-reversals is $\binom{n-1}{(n-1)/2}$ and the number of distinct arrangements inequivalent under rotations and reversals is N_0 where

$$N_0 = \frac{1}{2}\left(N(R) + \binom{n-1}{(n-1)/2}\right). \tag{4}$$

Combining equations (3) and (4), we have that N, the number of distinct "center empty" board configurations with A having the next move (and which are inequivalent under rotations and reversals), for any n is:

$$N = \frac{1}{2}\left(N(R) + \binom{2[n/2]}{[n/2]}\right). \tag{5}$$

We have, finally, the total number T of possible configurations: $4N(AA)$ for the center filled, N for the center empty with A having the next move, and N for the center empty with B having the next move. The total T is:

$$T = 2\left(\binom{2n-1}{n} + \binom{n-1}{[n/2]}\right) + \binom{2[n/2]}{[n/2]} + \frac{1}{2n}\sum_{d|n}\phi(d)\binom{2n/d}{n/d}. \tag{6}$$

Of these, from equation (1), the number of winning configurations is:

$$W = \begin{cases} \binom{2n-3}{n} + \binom{n-2}{[n/2]} & \text{for } n \geqslant 3 \\ 0 & \text{for } n = 1,2. \end{cases} \tag{7}$$

$W/2$ is the number of winning configurations for A and $W/2$ is the number of winning configurations for B. Of the total T, W of the configurations can never be attained in play.

Another way to arrive at T would be through the use of Pólya's enumeration theory [**6**]. Still another approach, specific to circular arrangements, can be found in [7].

Mu torere: $n = 4$

The formulas (6) and (7), of course, give the results we previously discussed for $n = 1, 2, 3$. We can now see the values for $n = 4$ which is *mu torere*. There are 92 configurations possible but 6 can never be attained. Of the 86 that can be attained, each player has 3 winning configurations. A list of all possible configurations is shown in FIGURE 6 and FIGURE 7 shows the flow of the game. The complexity is far greater than for $n = 3$. The game begins at (1) or (47). Assuming that A

(1) *AOAAAABBBB*	(24) *AAOABBBAAB*	(47) *BOBBBBAAAA*	(70) *BBOBAAABBA*
(2) *AOAAABABBB*	(25) *AAOABABBBA*	(48) *BOBBBABAAA*	(71) *BBOBABAAAB*
(3) *AOAAABBABB*	(26) *AAOABBABBA*	(49) *BOBBBAABAA*	(72) *BBOBAABAAB*
(4) *AOAABAABBB*	(27) *AAOABBBBAA*	(50) *BOBBABBAAA*	(73) *BBOBAAAABB*
(5) *AOAABBAABB*	(28) *ABOAAABBBA*	(51) *BOBBAABBAA*	(74) *BAOBBBAAAB*
(6) *AOAABABABB*	(29) *ABOBABAAAB*	(52) *BOBBABABAA*	(75) *BAOABABBBA*
(7) *AOAABABBAB*	(30) *ABOABBAABA*	(53) *BOBBABAABA*	(76) *BAOBAABBAB*
(8) *AOABABABAB*	(31) *ABOAABBABA*	(54) *BOBABABABA*	(77) *BAOBBAABAB*
(9) *AAOBBBAAAB*	(32) *ABOBBABAAA*	(55) *BBOAAABBBA*	(78) *BAOAABABBB*
(10) *AAOAABBBAB*	(33) *ABOABAAABB*	(56) *BBOBBAAABA*	(79) *BAOBABBBAA*
(11) *AAOBABAABB*	(34) *ABOBABABAA*	(57) *BBOABABBAA*	(80) *BAOABABABB*
(12) *AAOBAABBAB*	(35) *ABOABABAAB*	(58) *BBOABBAABA*	(81) *BAOBABABBA*
(13) *AAOBABABBA*	(36) *ABOBAAABBA*	(59) *BBOABABAAB*	(82) *BAOABBBAAB*
(14) *AAOAABABBB*	(37) *ABOAAABBAB*	(60) *BBOBBABAAA*	(83) *BAOBBBAABA*
(15) *AAOABABABB*	(38) *ABOAABABBA*	(61) *BBOBABABAA*	(84) *BAOBBABAAB*
(16) *AAOBBBAABA*	(39) *ABOBAABAAB*	(62) *BBOAAABBAB*	(85) *BAOABBABBA*
(17) *AAOBBBBAAA*	(40) *ABOAABAABB*	(63) *BBOAAAABBB*	(86) *BAOBBABBAA*
(18) *AAOBBAAABB*	(41) *ABOAABBBAA*	(64) *BBOAABBBAA*	(87) *BAOBBAAABB*
(19) *AAOBABABAB*	(42) *ABOBAAAABB*	(65) *BBOABABABA*	(88) *BAOABBBBAA*
(20) *AAOABABBAB*	(43) *ABOAAAABBB*	(66) *BBOBABAABA*	(89) *BAOBBBBAAA*
(21) *AAOBBABBAA*	(44) *ABOABABABA*	(67) *BBOAABAABB*	(90) *BAOBABABAB*
(22) *AAOABBAABB*	(45) *ABOAABBAAB*	(68) *BBOBAABBAA*	(91) *BAOBBAABBA*
(23) *AAOBAABABB*	(46) *ABOBABAABA*	(69) *BBOABBABAA*	(92) *BAOABABBAB*

FIGURE 6. **The ninety-two combinatorially possible configurations for *Mu torere*. The game begins with configuration (1), if *A* goes first, and with configuration (47), if *B* goes first. The winning configurations are (75), (85), and (88) for *A* and (29), (39), and (42) for *B*. Configurations (25), (26), (27), (71), (72), and (73) can never be attained during play.**

moves first, we examine some sequences of play. If moving inner pieces on the first move were not enjoined, A could win immediately by going from (1) to (88). And, if there were no restriction on B's second move, B could win in the short sequence (1) → (89) → (9) → (48) → (29). Some writers [**1**], [**8**] use a seven-move sequence leading to a win for A to illustrate the game: (1) → (89) → (9) → (48) → (33) → (70) → (3) → (85). Another [**3**] uses the same first six moves but concludes with a win for B: (3) → (87) → (9) → (48) → (29). These ignore the illegality of the

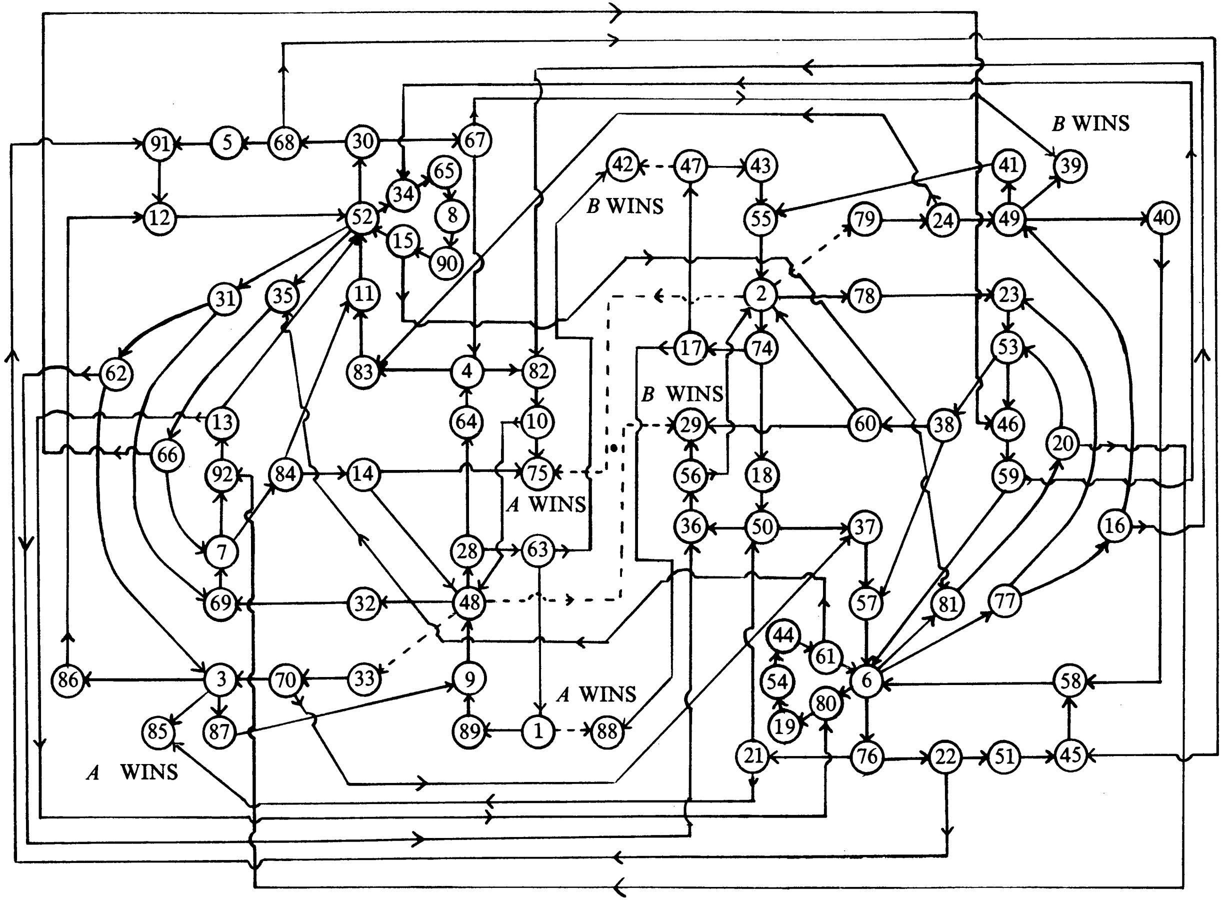
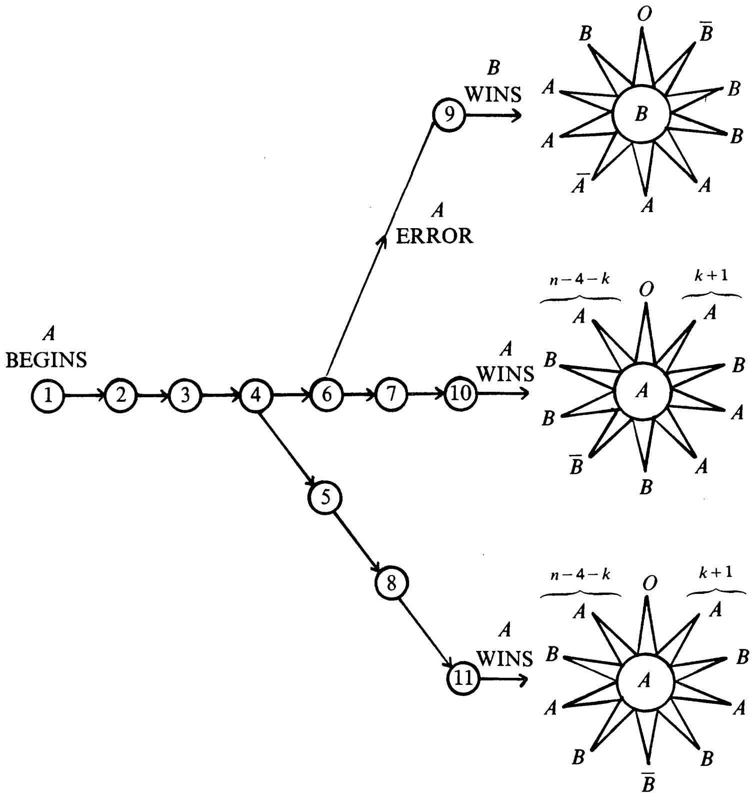


FIGURE 7. **Play for *Mu torere* starts at (1) if *A* begins or at (47) if *B* begins. (A dotted path is not permitted as a player's first or second move.) Configurations which are symmetric with respect to the center point of the diagram are complements.**

move from (48) to (33) by B as a second move. As just noted, if B could, in fact, move an inner marker, the preferable move would be (48) → (29) as it is an immediate win. However, as a second move for B, only (48) → (28) or (48) → (32) are possible. The former would not be wise as A could then easily win with (28) → (63) → (1) → (88). Even using the latter, B must soon again exercise caution as (32) → (69) → (7) → (84) → (14) → (75) is a win for A. A must be cautious as well. For example, choosing (31) → (62) rather than (31) → (69) would lead to a win for B, or choosing (4) → (83) rather than (4) → (82) misses winning the game in a few more moves. There are, for either player, short routes to winning against an inexperienced opponent but, with two careful players, the game can go on indefinitely. The many points of choice and multiple winning configurations make for great variety and require skill to avoid approaching pitfalls while planning how to entrap one's opponent.

The game for $n \geqslant 5$

Going beyond $n = 4$ to $n = 5$, the complexity increases as there are 272 attainable configura-



(1) $AOA\bar{A}AAABB\bar{B}BB$	(5) $ABOAA\bar{A}ABAB\bar{B}B$	(8) $BBOA\bar{A}ABAB\bar{B}BA$
(2) $BAOBB\bar{B}BBAA\bar{A}A$	(6) $ABOAB\bar{B}BBAA\bar{A}A$	(9) $BBOB\bar{B}BBAA\bar{A}AA$
(3) $AAOB\bar{B}BBAA\bar{A}B$	(7) $BBOAAB\bar{B}BBAA\bar{A}$	(10) $AOAB\bar{B}BBAA\bar{A}BA$
(4) $BOAB\bar{B}BBAA\bar{A}AB$		(11) $AOABAB\bar{B}BABA\bar{A}$

FIGURE 8. **The game for $n \geqslant 5$ can be easily won in a short sequence of moves. ($\bar{A} = n - 4$ consecutive A's; $\bar{B} = n - 4$ consecutive B's; $0 \leqslant k \leqslant n - 5$; e.g., for $n = 7$ the star would have 14 points, $\bar{A} = 3$ consecutive A's, $\bar{B} = 3$ consecutive B's, and on the winning stars the bracketed groups of $k + 1$ consecutive A's would contain 1, 2, or 3 A's while the bracketed groups of $n - 4 - k$ consecutive A's would contain 3, 2, or 1 A's.)**

tions with each player having 12 winning configurations. But the game becomes *far* less interesting because, if the first player is careful and makes no errors, he will win on his fourth move. If the first player is careless, the other player will win on his third move. This is not only true for $n = 5$ but for all $n \geqslant 5$ as is shown in FIGURE 8. *Mu torere* ($n = 4$), therefore, is the most enjoyable version of the game.

An 1856 account of *mu torere* describes the intensity of the players and the deep interest of the crowd of onlookers. Also, it reports that no foreigner who tried was able to win against a Maori player [**9**]. Another book, written more than one hundred years later, includes *mu torere* as one of the few games that still persists among the Maori despite the vast upheavals that have taken place in their culture due to the overwhelming influx of Europeans and European culture [**1**]. Our discussion has added another perspective on the game. There are many more questions that can be posed and answered about *mu torere*. Above all, it is a game from another culture that we can experience with enjoyment.

I am grateful to the referees for their careful reading of the manuscript and their thoughtful suggestions. In particular, I thank Richard K. Guy for calling my attention to [7] and for his helpful contribution in the case $n \geqslant 5$.

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Building the Cheapest Sandbox: an Allegory with Spinoff

WILLIAM P. COOKE
West Texas State University
Canyon, TX 79016

Allegory

The department secretary hands me a telephoned message from Rocky's Sand and Gravel. "I need a rectangular-based box that will hold three yards of sand when filled level. The width must be half the length, and the box is to be open at the top. Material for the base of the box costs $4 per square foot while the material for the sides and ends costs $3 per square foot. Can you give me the dimensions of the minimum-cost box? Thanks. (signed) Rocky."

It's the day before Easter recess, so hardly anyone has come to my calculus class. The four attending students and I agree to make a field trip to Rocky's Sand and Gravel. Rocky invites us inside his tiny office, graciously proffering coffee and cookies (sand tarts, naturally). We begin to show him how to formulate his problem using L, W, and H for the length, width, and height in feet.

We draw a box and observe that the area of the base will be LW square feet. There are four other pieces: two ends each with area HW square feet and the two longer sides with areas HL square feet. Incorporating costs, we get the objective function:

$$\begin{aligned}\text{Cost in dollars} = C &= 4LW + 3(2HW + 2HL)\\ &= 4LW + 6H(L+W).\end{aligned}$$

After Rocky informs us that "three yards" means 81 cubic feet, we write down the constraints:

$$LWH = 81 \quad \text{and} \quad W = 0.5L.$$

Substitution gives the cost as a function of length alone:

$$C(L) = 2L^2 + 1{,}458(1/L).$$

Setting dC/dL equal to zero yields

$$L = 9\cdot 2^{-1/3}, \qquad W = 9\cdot 2^{-4/3}, \qquad H = 2^{5/3},$$

which to five-decimal accuracy gives the dimensions in feet:

$$L = 7.14330, \qquad W = 3.57165, \qquad H = 3.17480.$$

The cost of the box is

$$\min C = 243\cdot 2^{1/3},$$

which to the nearest cent is

$$\min C = \$306.16.$$

This solution is duly reported to Rocky, who merely shakes his head slowly and chortles, "Heck, while you eggheads were formulating your model, I got to wondering about carrying that box in the bed of my pickup. A seven-by-three-and-a-half-foot base works nicely, which puts the height, as near as I'll be able to measure it with my tape, at about three feet, three and eleven-sixteenths inches." He hands us back our page of equations, derivatives and arithmetic, grinning broadly. "By the way, the cost of my box is $306.36, so your expensive educations have just saved me the grand sum of twenty cents—provided, of course, I find someone who can saw a board to the length of 7.1433 feet."

I do a quick calculation, dividing 81 by 7 times 3.5. Rocky should use 3.30612 feet for the height of his box, but since, then, his cost would be only $306.29, I decide to remain silent. After all, we would still beat him only by thirteen cents.

Our class piles into the station wagon and leaves the sand and gravel yard, secure in the knowledge that Rocky had failed to grasp the significance of the exercise. A freshman tentatively ventures: "Or have we?" She is suitably contrite after the rest of us point out that if Rocky were to manufacture 20,000 such boxes we could save him $4,000.

Upon my return to the office I am handed another message from Rocky. We have made a mistake. Our dimensions work just fine for the schematic diagram we had drawn on paper, but he has to have some *overlap* to be able to nail the box together.

Rocky's message says the base is two inches thick while the boards for the sides and ends are one inch thick. The base is to fit inside the sides and ends, with the end pieces just as wide as the base board and extending to the bottom of the base. The side pieces also extend to the bottom of the base but are each to be two inches longer than the base board to allow for a one-inch overlap on each end for nailing to the end pieces.

I reformulate the problem letting L, W, and H be the *interior* dimensions of the box. Then the area of the base is still LW square feet, but each side has area $(L+1/6)(H+1/6)$ and each end has area $W(H+1/6)$ square feet. Now the objective function is:

$$C=4LW+6(H+1/6)(L+W+1/6),$$

with constraints

$$LWH=81 \quad \text{and} \quad W+1/6=(1/2)(L+1/6).$$

I decide to express C as a function of L as I did before. In this case,

$$C(L)=(1/3)L(6L-1)+(1/12)[108L^3-12L^2+104{,}975L+5{,}832]/[L(6L-1)].$$

I call my wife to say I'll be late for dinner.

Differentiating, setting equal to zero, and clearing of fractions and common factors, I find the following quintic equation:

$$864L^5-36L^4-60L^3-314{,}921L^2-34{,}992L+2{,}916=0.$$

Trial and error with a calculator, starting in the neighborhood of 7, gives L to five places, with substitution into the constraints giving W and H:

$$L=7.19655, \qquad W=3.51494, \qquad H=3.20215.$$

The cost of the box is

$$\min C=\$321.06.$$

By the way, the quintic equation, with two sign changes, has to have another positive root, in this case $L=0.055556$. But this solution gives the box negative interior width and negative height, and is the location of a relative maximum.

I call Rocky to tell him the new solution, but before I get a chance to say anything, he tells me that he decided to just add in the required overlaps to his original solution, except that he is now using $H=3.30612$ so I can understand his answer. The cost seemed reasonable, so he has already built the box.

Smugly, I inquire: "What's the cost of your box?"

"$320.26," he replies.

"I'll call you back."

He has beaten me by 80 cents. Moreover, his box holds 80.999940 cubic feet while mine would only hold 80.999798. But soon I discover the discrepancy. The outside dimensions of Rocky's box are $L+1/6=7.16667$ feet and $W+1/6=3.66667$ feet. Half of 7.16667 is 3.583335. I call him again.

"I'm sorry to have to tell you this, Rocky, but your box has violated a constraint. Its width is over eight-hundredths of an inch greater than half its length."

"Who cares?" says Rocky, hanging up.

That's odd. He didn't even ask for my optimal solution.

Spinoff

This story illustrates the difference in viewpoints of an academician and a practical man. In its first, simpler version the problem also provides a nice example of a function that has a very flat graph in the neighborhood of its minimum value. If you were to graph $C(L)$ from the second version, however, the formulation that allows for overlap, you would see a flattened branch to the right of $L = 1/6$ and another branch with a pronounced peak between $L = 0$ and $L = 1/6$. That second function has no absolute minimum, but its relative minimum is the optimal practical solution. Thus the problem could be a nice classroom example to exhibit how a minor change in formulation can lead to a major jump in level of difficulty: not conceptual difficulty, of course, only algebraic. It just requires the usual solution: the first derivative equal to zero and solve for a stationary point.

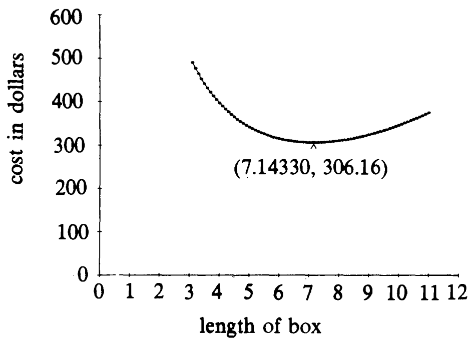

FIGURE 1

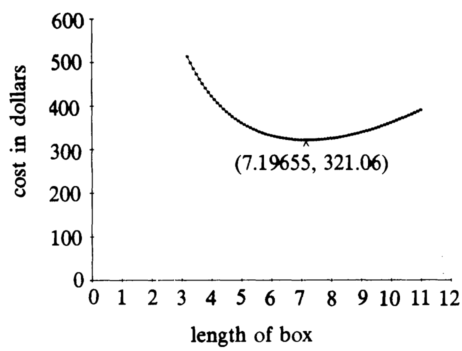

FIGURE 2

But setting first derivatives equal to zero and testing to see what kind of stationary point has been found is not the whole story in optimization. That approach usually works just fine for unconstrained functions of a single variable. Restricting the domain, however, often leads to an end-point solution where the first derivative is not zero.

With multivariable functions, setting all first partials equal to zero and attempting to solve the resulting system often presents complications. The computational problem of simply finding a stationary point can be formidable, and even if a solution for the system may be found, the resulting point might fail the test for a relative extremum: for example, positive semidefiniteness of the Hessian matrix of second partials in the case of a minimum. Further, since multivariable problems arising in practical situations almost always involve constraints on the variables, boundary-point relative extrema would have to be compared with any interior stationary points.

When the constraints are equations we can resort to Lagrange multipliers, but here again computational difficulties usually arise. When the constraints are inequalities, however, the classical methods of the calculus become less useful. Here we often must appeal to an iterative mathematical programming algorithm. Since a tutorial in mathematical programming would be inappropriate here, suffice it to say that one fervently hopes for convex feasible regions and convex or concave objective functions, where global optimality can be assured.

Students without much practical experience in optimization tend to believe that the classical calculus is the universal panacea. Somehow the objective of the problem seems to be finding the derivative, not finding the extremum. They tend to differentiate blithely and set equal to zero without first stopping to think about the problem. Here is an example that might be used to help them break that habit.

A standard problem in statistics is the maximum-likelihood estimation of a parameter. Basically, the rationale is to obtain a sample and then ask what value of the parameter would have made the sample we obtained most probable amongst the set of all possible values of the parameter. The approach involves forming the joint probability distribution (or density) of the sample values, treating that as a function of the parameter for fixed sample values (the likelihood function), and then finding the maximum of that function. If the parameter of interest is constrained, or if it can assume only a finite or discrete set of values, one can get into trouble fast merely using a calculus approach.

Suppose we are told that an urn contains five marbles, and that some of them may be white and some of them may be red. Our objective is to estimate the proportion of red marbles in the urn, p, where, of course, the only possible values for p are 0, 0.2, 0.4, 0.6, 0.8, and 1.0. We take a sample of size $n=3$, drawing with replacement, from the urn, and we observe one red and two white marbles in the sample. What is the maximum-likelihood estimate of p?

The likelihood function in this case is binomial with parameters $n=3$ and p, where p is the probability of obtaining a red marble on one draw. This function, call it $L(p)$, is:

$$L(p)=3p^1(1-p)^2.$$

Simply differentiating and setting equal to zero gives

$$3p^2-4p+1=0,$$

which has solutions $p=1/3$ and $p=1$. Of course neither of these is a permissible estimate of p, since $1/3$ (by far the most popular wrong answer) is not in the domain of L and our observation of two white marbles in the sample rules out $p=1$.

The correct maximum-likelihood estimate of p is 0.4, which is the value that makes the sample we observed most probable over the finite set of possibilities for p. To observe that result, merely evaluate $L(p)$ at $p=0, 0.2, 0.4, 0.6, 0.8$, and 1.0. Students given this problem, unless forewarned, are prone to overlook the discrete domain of the function and differentiate themselves into an impossible answer.

Historical note

The physical setting of the allegory was suggested to the author by a story related to him by his father, who was for a while one of those notorious "coaches who teach math." A contractor had called him to ask for dimensions for a hopper that would have a truncated, four-sided pyramidal funnel and a rectangular-box top portion, and that had to hold exactly a yard of sand. Dad's solution? "Build the funnel, then just keep adding height to the box until the hopper holds a yard of sand." Score one for the coaches.

The author is happy to acknowledge contributions of two reviewers of this paper. One suggested the title and made helpful editorial changes. The other suggested adding material on optimization beyond just the basic allegory.

Products of Sines and Cosines

STEVEN GALOVICH
Carleton College
Northfield, MN 55057

In [3], Z. Usiskin presents a number of intriguing identities involving products of specific values of the sine function. Here are some examples:

$$\sin 10° \sin 50° \sin 70° = \frac{1}{8}, \tag{1}$$

$$\sin 6° \sin 42° \sin 66° \sin 78° = \frac{1}{16}. \tag{2}$$

Most of the identities derived by Usiskin have the same form: The product of the sines of k appropriately chosen angles (k is a positive integer), all between 0° and 90°, equals $1/2^k$.

In a recent conversation, Usiskin asked for an "explanation" of these identities. By that he meant more than merely proofs of these equations. His paper contains proofs of (1) and (2) and several other similar formulas. Rather, Usiskin is asking for a general statement which contains his identities as special cases and for both rigorous and heuristic arguments which justify the generalization. This note provides this type of explanation.

We begin by writing the left side of (1) and (2) in terms of complex roots of unity. We then analyze generalizations of the expressions thus obtained and we present two evaluations of the generalized expressions. Both are based on number-theoretic considerations; the first is elementary in nature, while the second, involving ideas from algebraic number theory, is more sophisticated. The second approach also provides the heuristic "explanation" sought by Usiskin for the identities. The principal result of the paper is Theorem 2, which evaluates the products that generalize identities (1) and (2). Theorem 1, a well-known fact of algebraic number theory, is used to prove both Theorem 2 and Theorem 3. The latter theorem describes an interesting property of regular n-gons inscribed in a unit circle.

Formulation of the problem

To set the stage for our formal argument, let us play with the first identity. We begin by rewriting it as

$$(2\sin 10°)\,(2\sin 50°)\,(2\sin 70°) = 1. \tag{3}$$

In the field of complex numbers,

$$[(\cos 10° + i\sin 10°) - (\cos 10° - i\sin 10°)]/i = 2\sin 10°.$$

Let $\zeta = \cos 10° + i\sin 10°$. The complex conjugate and inverse of ζ is $\zeta^{-1} = \cos 10° - i\sin 10°$ and

$$2\sin 10° = (\zeta - \zeta^{-1})/i.$$

Also $2\sin 50° = (\zeta^5 - \zeta^{-5})/i$ and $2\sin 70° = (\zeta^7 - \zeta^{-7})/i$. In terms of ζ and ζ^{-1}, equation (3) becomes:

$$\prod_{j=1,5,7} (\zeta^j - \zeta^{-j})/i = 1.$$

Note that by De Moivre's formula, $\zeta^{36} = 1$, i.e., ζ is a 36th root of unity.

Through analogous manipulations, identity (2) reduces to

$$\prod_{j=1,7,11,13} (\alpha^j - \alpha^{-j})/i = 1, \tag{4}$$

where $\alpha = \cos 6° + i \sin 6°$. By the way, note another identity:

$$\prod_{j=1,7,11,13} 2\cos(6j)° = \prod_{j=1,7,11,13} (\alpha^j + \alpha^{-j}) = 1. \tag{5}$$

Also observe that α is a 60th root of unity.

Lest we develop the impression that products of these kinds always equal 1, consider

$$(2\cos 10°)(2\cos 50°)(2\cos 70°) = \sqrt{3},$$

an interesting identity in its own right.

What is the general pattern which describes the values of these products? Several of the identities presented by Usiskin, including (3) and (4), can be rewritten in the following form: For $n \in \{36, 60, 72, 120, 130, 360\}$ and $r = 2\pi/n$ (henceforth we use radian measure)

$$\prod_{j=1}^{n/4}{}'(2\sin(j \cdot r)) = 1 \tag{6}$$

where $\prod_j{}'$ denotes the product of those integers j that are relatively prime to n. When $n = 36$ and $n = 60$, one obtains identities (3) and (4), respectively.

Our overall goal (accomplished in Theorem 2) is to describe products of the following form:

$$\prod_{j=1}^{k}{}' 2f(jr) \tag{7}$$

where $f(x) = \sin x$ or $f(x) = \cos x$ and $k = n/4$, $n/2$, or n. As in (4) and (5), these numbers can be represented as products of sums of roots of unity. Thus we turn to a study of basic arithmetic properties of roots of unity (i.e., those involving addition, subtraction, and multiplication).

Roots of unity

The investigation of arithmetic properties of roots of unity, the so-called theory of cyclotomic fields, was pioneered in the mid-19th century by the number theorist E. E. Kummer. The facts about cyclotomic fields which we need are elementary and can be established without recourse to the beautiful and intricate machinery constructed by Kummer. Later we shall derive the key results using ideas and results of algebraic number theory.

Let n be a positive integer such that $n \neq 1, 2, 4$. Let

$$\zeta_n = \cos(2\pi/n) + i\sin(2\pi/n);$$

ζ_n is a primitive nth root of unity, which means that $\zeta_n^n = 1$ and $\zeta_n^j \neq 1$ for $1 \leq j \leq n-1$. Of

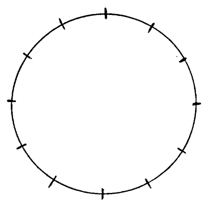

FIGURE 1. **The 12th roots of unity in the complex plane.**

course, for $1 \leq j \leq n-1$, ζ_n^j is also an nth root of unity, hence is a root of the equation $x^n - 1 = 0$. Notice that

$$\zeta_n^j + \zeta_n^{-j} = 2\cos(2\pi j/n)$$

and

$$\left(\zeta_n^j - \zeta_n^{-j}\right)/i = 2\sin(2\pi j/n).$$

The primitive nth roots of unity are ζ_n^j for $(j, n) = 1$ where (j, n) is the greatest common divisor of j and n. There are $\varphi(n)$ primitive roots of unity where $\varphi(n)$, the Euler phi-function, is defined to be the number of positive integers less than n that are relatively prime to n. The well-known formula for $\varphi(n)$ will come in handy later: Write $n = p_1^{a_1} \cdots p_m^{a_m}$ where $p_1, \cdots, p_m$ are distinct primes. Then

$$\varphi(n) = \prod_{i=1}^{m} p_i^{a_i - 1}(p_i - 1).$$

Over the field $\mathbb{C}$ of complex numbers, we have the following polynomial factorizations:

$$x^n - 1 = \prod_{j=0}^{n-1}\left(x - \zeta_n^j\right)$$

and

$$(x^n - 1)/(x - 1) = 1 + x + \cdots + x^{n-1} = \prod_{j=1}^{n-1}\left(x - \zeta_n^j\right). \tag{8}$$

When $x = 1$ is substituted in (8), the result is the following factorization of n in $\mathbb{C}$:

$$\prod_{j=1}^{n-1}\left(1 - \zeta_n^j\right) = n. \tag{9}$$

We wish to investigate the product

$$\rho_n = \prod_{j=1}^{n-1}{}'\left(1 - \zeta_n^j\right), \tag{10}$$

since it is closely related to the products of sines and cosines described in (7). To determine the value of ρ_n, we use a standard tactic in number theory: First consider the case in which n is prime, then allow n to be arbitrary and use mathematical induction on the number of prime factors of n.

Suppose n is prime. Then $\rho_n = \prod_{j=1}^{n}(1 - \zeta_n^j) = n$. Next, to get a feel for the general case, suppose $n = p^2$ where p is prime; then from (9) and the fact that $\zeta_{p^2}^p = \zeta_p$, it follows that

$$p^2 = \rho_n \prod_{k=1}^{p-1}\left(1 - \zeta_n^{pk}\right) = \rho_n \prod_{k=1}^{p-1}\left(1 - \zeta_p^k\right) = \rho_n \cdot \rho_p = \rho_n \cdot p.$$

Thus $\rho_n = p$, if $n = p^2$.

We now have both the basis step and the gist of the idea for an inductive proof of the next result.

THEOREM 1. *If* ρ_n *is defined by* (10), *then*

$$\rho_n = \begin{cases} p & \text{if } n = p^a \text{ where } p \text{ is prime} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. We argue by induction on the number, f, of prime factors of n. (We count repeated prime factors, so 12 has 3 prime factors by our reckoning.) We have already established the basis step, $f = 1$. Let n be an arbitrary positive integer with f factors. Suppose the theorem holds for all positive integers with fewer than f factors. (Writing $n = p_1^{a_1} \cdots p_m^{a_m}$ where the p_i are distinct primes, we have $f = \sum_{j=1}^{m} a_i$.) Then

$$n = \prod_{j=1}^{n} (1 - \zeta_n^j) = \rho_n \prod_{\substack{j=1 \\ (j,n)>1}}^{n} (1 - \zeta_n^j) = \rho_n \prod_{\substack{d|n \\ d\neq 1}} A_d$$

where the last product is taken over divisors on n that are greater than 1 and for each such divisor d,

$$A_d = \prod_{(j,n)=d} (1 - \zeta_n^j).$$

We claim that $A_d = \rho_{n/d}$. Assuming this claim, the proof of the theorem proceeds as follows:

Case 1. Suppose $m = 1$; in other words, $n = p^a$ where p is prime. Then

$$\prod_{\substack{d|n \\ d\neq 1}} A_d = \prod_{\substack{d|n \\ d\neq 1}} \rho_{n/d} = \prod_{i=i}^{a-1} p = p^{a-1}$$

by inductive hypothesis, hence $\rho_n = n/p^{a-1} = p$.

Case 2. Suppose $m > 1$. Let $S_i = \{n/p_i, n/p_i^2, \dots, n/p_i^{a_1}\}$ for $1 \le i \le m$. If $d \in S_i$, then $\rho_{n/d} = p_i$ by induction and

$$\prod_{d \in S_i} A_d = \prod_{d \in S_i} \rho_{n/d} = p_i^{a_i}.$$

If

$$d \notin \bigcup_{i=1}^{m} S_i,$$

then $A_d = \rho_{n/d} = 1$ by induction. Therefore,

$$n = \rho_n \prod_{\substack{d|n \\ d\neq 1}} A_d = \rho_n \cdot n \quad \text{and} \quad \rho_n = 1.$$

We now must prove the claim that $A_d = \rho_{n/d}$:

$$\begin{aligned} A_d &= \prod_{(j,n)=d} (1 - \zeta_n^j) \\ &= \prod_{l=1}^{n/d}{}' \left(1 - (\zeta_n^d)^l\right) \\ &= \prod_{l=1}^{n/d}{}' \left(1 - \zeta_{n/d}^l\right) = \rho_{n/d}. \end{aligned}$$

Products of sines and cosines

Let us turn to the evaluation of products of sines and cosines. Let n be a positive integer different from 1, 2, and 4. For $s = 1, 1/2, 1/4$, and with $r = 2\pi/n$, let

$$C_n(s) = \prod_{j=1}^{ns}{}' 2\cos(jr)$$

and

$$S_n(s) = \prod_{j=1}^{ns}{}' 2\sin(jr).$$

Consider first $C_n(1)$:

$$C_n(1) = \prod_{j=1}^{n}{}' (\zeta_n^j + \zeta_n^{-j})$$

$$= \prod_{j=1}^{n}{}' \zeta_n^{-j}(1+\zeta_n^{2j})$$

$$= \zeta_n^{-J} \prod_{j=1}^{n}{}'(1+\zeta_n^{2j})$$

where

$$J = \sum_{j=1}^{n}{}' j,$$

the summation taken over integers j such that $(j, n) = 1$. On the other hand:

$$S_n(1) = \prod_{j=1}^{n}{}'(\zeta_n^j - \zeta_n^{-j})/i$$

$$= i^{-\varphi(n)}\zeta_n^{-J} \prod_{j=1}^{n}{}'(\zeta_n^{2k} - 1).$$

As an exercise the reader may check that $\varphi(n)$ is even and $J = n\varphi(n)/2$. Thus $i^{-\varphi(n)} = (-1)^{\varphi(n)/2}$ and $\zeta_n^{-J} = 1$, implying that

$$C_n(1) = \prod_{j=1}^{n}{}'(1+\zeta_n^{2j})$$

and

$$S_n(1) = (-1)^{\varphi(n)/2} \prod_{j=1}^{n}{}'(\zeta_n^{2j} - 1) = (-1)^{\varphi(n)/2} \prod_{j=1}^{n}{}'(1 - \zeta_n^{2j}).$$

The numbers $C_n(1)$ and $S_n(1)$ look very similar to ρ_n. To establish the exact relationship, we consider several cases.

Case 1. $n = p^a$ where p is an odd prime. Because n is odd, the set $\{\zeta_n^{2j} | 1 \le j \le n, (j, n) = 1\}$ is precisely the set of primitive nth roots of unity, while the set $\{-\zeta_n^{2j} | 1 \le j \le n, (j, n) = 1\}$ is the set of primitive $(2n)$th roots of unity. Therefore,

$$C_n(1) = \prod_{j=1}^{n}{}'(1 - (-\zeta_n^{2j})) = \rho_{2n} = 1$$

and

$$S_n(1) = (-1)^{\varphi(n)/2} \prod_{j=1}^{n}{}'(1 - \zeta_n^{2j})$$

$$= (-1)^{\varphi(n)/2}\rho_n = (-1)^{\varphi(n)/2} p.$$

Case 2. $n = 2^a$. The sets $\pm\{-\zeta_n^{2j} | 1 \le j \le n, (j, n) = 1\}$ each consist of the (2^{a-1})st roots of unity with each such root appearing twice. Thus,

$$C_n(1) = \prod_{j=1}^{n}{}'(1 - (-\zeta_n^{2j}))$$

$$= \prod_{j=1}^{n}{}'(1 - \zeta_n^{2j}) = S_n(1)$$

$$= \prod_{j=1}^{n/2}{}'(1 - \zeta_{n/2}^{j})^2$$

$$= 2^2 = 4.$$

Case 3. $n=2p^a$ where p is an odd prime. As in case 1, $C_n(1)=1$ and $S_n(1)=p$. The arguments parallel those in case 1.

Case 4. $n=4p^a$ where p is an odd prime. This time $\{-\zeta_n^{2j} | 1 \le j \le n, (j,n)=1\}$ is the set of p^ath roots of unity with each such root appearing twice. Therefore

$$C_n(1) = \rho_{p^a}^2 = p^2$$

and

$$S_n(a) = \rho_{2p^a}^2 = 1.$$

Case 5. All other cases. In all other instances, $C_n(1) = S_n(1) = 1$. For example if n is odd, then n is not a prime power and $C_n(1) = S_n(1) = \rho_n = 1$. If n is even, then neither $n/2$ nor $n/4$ is a prime power and $C_n(1) = S_n(1) = 1$. We leave the details for the reader.

Next we consider $C_n(s)$ and $S_n(s)$ for $s=1/2$ or $1/4$. Since $\varphi(n)$ is even, $|C_n(1)| = C_n(1/2)^2$ and $|S_n(1)| = S_n(1/2)^2$. If $\varphi(n)$ is divisible by 4 (this occurs, for example, in case 1 for $p \equiv 1 \pmod 4$) and in cases 2, 4, and 5), then $|C_n(1/2)| = C_n(1/4)^2$ and $S_n(1/2) = C_n(1/4)^2$. Note that $S_n(1/4)$, $C_n(1/4)$ and $S_n(1/2)$ are always positive. The sign $C_n(1/2)$ is somewhat subtle since $\cos(x) < 0$ if $\pi/2 < x \le \pi$. We carry out the analysis in one case.

Suppose $n = p^a$ where p is an odd prime. Then

$$C_n(1/2) = \prod_{j=1}^{n/2}{}' 2\cos(jr)$$

has the same sign as $(-1)^b$ where b is the number of integers j such that $\pi/2 < jr < \pi$, which equals the number of integers j such that $p^a/4 < j < p^a/2$. A straightforward calculation shows that this number is even if $p \equiv 1,7 \pmod 8$ and odd if $p \equiv 3,5 \pmod 8$. Thus

$$C_n(1/2) = \begin{cases} 1 & \text{if } p \equiv 1,7 \pmod 8 \\ -1 & \text{if } p \equiv 3,5 \pmod 8. \end{cases}$$

We summarize our findings in the following theorem.

THEOREM 2. *Suppose $n \ne 1,2,4$. The values of $C_n(s)$ and $S_n(s)$ are given in* TABLE 1.

n	s	$S_n(s)$	$C_n(s)$
1. $n=p^a$	1	$(-1)^{\varphi(n)/2}p$	1
p an odd prime	1/2	$\sqrt{p}$	1 if $p^a \equiv 1,7 \pmod 8$ -1 if $p^a \equiv 3,5 \pmod 8$
2. $n=2^a, a>2$	1	4	4
	1/2	2	2 if $a>3$ -2 if $a=3$
3. $n=2p^a$	1	$(-1)^{\varphi(n)/2}p$	1
p an odd prime	1/2	$\sqrt{p}$	1 if $p^a \equiv 1,3 \pmod 8$ -1 if $p^a \equiv 5,7 \pmod 8$
4. $n=4p^a$	1	1	p^2
p an odd prime	1/2	1	p if $p^a \equiv 1 \pmod 4$ $-p$ if $p^a \equiv 3 \pmod 4$
	1/4	1	$\sqrt{p}$
5. All other cases	1	1	1
	1/2	1	1
(if 4 divides n)	1/4	1	1

TABLE 1

Algebraic interpretation

Thus far we have responded to part of Usiskin's challenge. By our standards an explanation of a phenomenon consists in part of formulating and proving a general statement which contains the empirical observation as a special case. But what are the heuristic notions from algebraic number theory lurking behind the theorems we have proved?

All the calculations used to prove Theorems 1 and 2 involve linear combinations of nth roots of unity with integer coefficients. The set of all complex numbers of this form constitutes a subring of $\mathbf{C}$, called the **ring of cyclotomic integers:**

$$\mathbf{Z}[\zeta_n] = \left\{ \sum_{j=0}^{m} a_j\zeta_n^j \,\middle|\, m \in \mathbf{Z}, m \geq 0, a_j \in \mathbf{Z} \right\}.$$

Actually, since $\zeta_n^n = 1$,

$$\mathbf{Z}[\zeta_n] = \left\{ \sum_{j=0}^{n-1} a_j\zeta_n^j \,\middle|\, a_j \in \mathbf{Z} \right\}.$$

It is not difficult to show that

$$\mathbf{Z}[\zeta_n] = \left\{ \sum_{j=0}^{n-2} a_j\zeta_n^j \,\middle|\, a_j \in \mathbf{Z} \right\}.$$

The ring $\mathbf{Z}[\zeta_n]$ is a subring of the **cyclotomic field** of nth roots of unity

$$\mathbf{Q}(\zeta_n) = \left\{ \sum_{j=0}^{n-1} a_j\zeta_j \,\middle|\, a_j \in \mathbf{Q} \right\},$$

which is an algebraic extension of $\mathbf{Q}$ of degree $\varphi(n)$. Each element α of $\mathbf{Q}(\zeta_n)$ is a root of a polynomial equation of the form $f_\alpha(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_1x + b_0$ where each b_i is a rational number. The elements of $\mathbf{Z}[\zeta_n]$ are precisely those elements α of $\mathbf{Q}(\zeta_n)$ for which the coefficients b_i of $f_\alpha(x)$ are integers. $\mathbf{Z}[\zeta_n]$ is called the **integral closure** of $\mathbf{Z}$ in $\mathbf{Q}(\zeta_n)$ and is the maximal (in the sense of inclusion) subring of $\mathbf{Q}(\zeta_n)$ which contains $\mathbf{Z}$ and which has finite rank as an abelian group.

Let R be a commutative ring with unity. An element u in R is a unit if there exists $v \in R$ such that $uv = 1$. The set of units $U(R)$ of a commutative ring forms a group under ring multiplication. For example, $U(\mathbf{Z}) = \{\pm 1\}$. In $\mathbf{Z}[\zeta_n]$, the roots of unity, $\pm\zeta_n^j$, $0 \leq j \leq n-1$, are all units. The only integers that are units in $\mathbf{Z}[\zeta_n]$ are ± 1. The structure of the full unit group of $\mathbf{Z}[\zeta_n]$ is given by the famous Dirichlet unit theorem: $U(\mathbf{Z}[\zeta_n])$ is the direct product of the set of units $\{\pm\zeta_n^j | 0 \leq j \leq n-1\}$ with $\varphi(n)/2 - 1$ copies of $\mathbf{Z}$. Theorem 1 implies that if n is not a prime power and $(j, n) = 1$, then $1 - \zeta_n^j$ is a unit in $\mathbf{Z}[\zeta_n]$.

Let x be a nonzero nonunit in R. Then x is called **irreducible** if whenever $x = yz$ for $y, z \in R$, either y or z is a unit of R. In $\mathbf{Z}$ the irreducible elements are precisely the prime numbers. As we shall soon see, if n is a prime power, then the element $1 - \zeta_n$ is irreducible in $\mathbf{Z}[\zeta_n]$.

We now introduce the norm function from $\mathbf{Q}(\zeta_n)$ to $\mathbf{Q}$. Let $a = \sum_{i=0}^{n-1} a_i\zeta_n^i \in \mathbf{Q}(\zeta_n)$. We define the polynomial $a(x) = \sum_{i=0}^{n-1} a_ix^i$. (Then $a = a(\zeta_n)$.) Define the complex number $N(a)$, called the **norm of a,** by

$$N(a) = \prod_{j=1}^{n}{}' a(\zeta_n^j).$$

From elementary Galois theory it follows that $N(a)$ is rational. Thus N maps $\mathbf{Q}(\zeta_n)$ into $\mathbf{Q}$. As a function N has the following properties.

1. If $a \in \mathbf{Z}[\zeta_n]$, then $N(a) \in \mathbf{Z}$.
2. For $a \in \mathbf{Q}(\zeta_n)$, $a \neq 0$, $N(a) > 0$.
3. For $a, b \in \mathbf{Q}(\zeta_n)$, $N(ab) = N(a)N(b)$. (In other words, N is a multiplicative function.)

Notice that if $u \in U(\mathbf{Z}[\zeta_n])$, then $N(u) = 1$. For if $uv = 1$ for some $v \in \mathbf{Z}[\zeta_n]$, then $1 = N(1) = N(uv) = N(u)N(v)$. Since $N(u)$ and $N(v)$ are both positive integers, $N(u) = 1$. Conversely, if $N(u) = 1$, then $u \in U(\mathbf{Z}[\zeta_n])$:

$$1 = N(u) = \prod_{j=1}^{n}{}' u(\zeta_n^j) = u(\zeta_n) \prod_{j=2}^{n}{}' u(\zeta_n) = uv \quad \text{where} \quad v = \prod_{j=2}^{n}{}' u(\zeta_n) \in \mathbf{Z}[\zeta_n].$$

Next we show that if $N(a) = q$ is prime in $\mathbf{Z}$, then a is irreducible in $\mathbf{Z}[\zeta_n]$. For if $a = bc$ for some $b, c \in \mathbf{Z}[\zeta_n]$, then $q = N(a) = N(b)N(c)$ which means that $N(b) = 1$ or $N(c) = 1$. If $N(b) = 1$ (or $N(c) = 1$), then b (or c) is a unit in $\mathbf{Z}[\zeta_n]$, hence a is irreducible.

We can now interpret Theorem 1 in the language of algebraic number theory. First consider ρ_n. When $n = p^a$ is a prime power, then $\rho_n = N(1 - \zeta_n) = p$. In this case $1 - \zeta_n$ (and $1 - \zeta_n^j$ for $(j, n) = 1$) is irreducible in $\mathbf{Z}[\zeta_n]$. When n is not a prime power, then $\rho_n = N(1 - \zeta_n) = 1$ and $1 - \zeta_n$ (and $1 - \zeta_n^j$ for $(j, n) = 1$) is a unit in $\mathbf{Z}[\zeta_n]$.

Now the numbers $C_n(1)$ and $S_n(1)$ are closely related to ρ_n. For example, the following table expresses $C_n(1)$ in terms of ρ_n.

n	$p^a, p \neq 2$	2^a	$2p^a$	$4p^a$	all other cases
$C_n(1)$	ρ_{2n}	$(\rho_{n/2})^2$	$\rho_{n/2}$	$(\rho_{n/4})^2$	ρ_n

If $C_n(1) = \rho_k^e$ where k is not a prime power and $e = 1$ or 2, then $C_n(1)$ is the norm of a unit of $\mathbf{Z}[\zeta_n]$, hence $C_n(1) = 1$. Since "most" positive integers are not prime powers, most of the time $C_n(1) = 1$ and $S_n(1) = 1$. The exceptional cases occur when n is a prime power or close to a prime power. Then $C_n(1)$ (or $S_n(1)$) is either the norm of an irreducible of $\mathbf{Z}[\zeta_n]$ or the square of the norm of an irreducible, hence $C_n(1)$ (or $S_n(1)$) is either prime or the square of a prime.

With these remarks the algebraic "explanation" of our principal result is complete. Readers wishing to dig deeper into cyclotomic fields and algebraic number theory may consult **[1]**, **[2]**, or **[4]**.

Chords of regular *n*-gons

As a final application of this circle of ideas, we establish a result about regular n-gons. Although the next theorem and proof are evidently well known, it is natural to include them in this note.

THEOREM 3. *Let P be a regular n-gon inscribed in a unit circle and let v be a fixed vertex of P. The product of the lengths of the $n - 1$ chords of P drawn from v to the other $n - 1$ vertices is n.*

Proof. Place the circle so that its center is $(0,0)$ and its vertices are $\{\zeta_n^j | 0 \leq j \leq n - 1\}$. Without

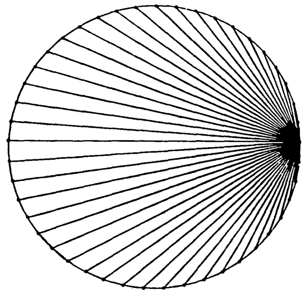

FIGURE 2

loss of generality, suppose $v=1$. For each j, $1 \leq j \leq n-1$, the vector joining ζ_n^j to 1 is represented by the complex number $1-\zeta_n^j$ and has length $|1-\zeta_n^j|$. The product of these lengths is

$$\prod_{j=1}^{n-1} |1-\zeta_n^j| = \left| \prod_{j=1}^{n-1} \left(1-\zeta_n^j\right) \right| = n.$$

I wish to thank D. Schattschneider, Z. Usiskin, and the referee for helpful comments.

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[2] K. Ireland and M. Rosen, A Classical Introduction to Modern Number Theory, Springer-Verlag, New York, 1982.

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[4] L. Washington, Introduction to Cyclotomic Fields, Springer-Verlag, New York, 1982.

PROBLEMS

Loren C. Larson, Editor
Bruce Hanson, Associate Editor
St. Olaf College

Proposals

To be considered for publication, solutions should be received by September 1, 1987

Correction to 1253 (December, 1986, p. 297) Replace the open interval $(0, 2)$ with the open interval $(0, 2 - \ln 2)$.

1262. *Proposed by Erwin Just, Bronx Community College, Bronx, New York.*

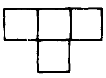
Show that it is possible to enumerate the rational numbers in the open interval $(0, 1)$ so that in their decimal expansions

$$\begin{aligned} r_1 &= .a_{11}a_{12}a_{13}a_{14}\dots \\ r_2 &= .a_{21}a_{22}a_{23}a_{24}\dots \\ r_3 &= .a_{31}a_{32}a_{33}a_{34}\dots \\ &\dots \end{aligned}$$

the "columns" $c_k = .a_{1k}a_{2k}a_{3k}a_{4k}\dots$ are rational for $k = 1, 2, 3, \dots$.

1263. *Proposed by Miklós Laczkovich, Eötvös Lorand University, Budapest, Hungary.*

A T-tetromino is a configuration of four unit squares arranged in the following shape:



a. Prove that if an $m \times n$ rectangular board can be tiled with T-tetrominoes, then the product mn is a multiple of eight.

*b. Prove or disprove: An $m \times n$ rectangular board can be tiled with T-tetrominoes if and only if m and n are multiples of four.

ASSISTANT EDITORS: Clifton Corzatt and Theodore Vessey, *St. Olaf College. We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren C. Larson, Department of Mathematics, St. Olaf College, Northfield, MN 55057.

1264. *Proposed by Daniel B. Shapiro and Bostwick Wyman, The Ohio State University, Columbus.*

a. Given $a_1,\ a_2 \in \mathbf{R}$ define the sequence $(a_k)_{k=1}^{\infty}$ by setting

$$a_k = \frac{a_{k-1} + a_{k-2}}{2}$$

for all $k \geqslant 3$. Prove that $L(a_1, a_2) = \lim_{k\to\infty} a_k$ exists and equals

$$\tfrac{1}{3}a_1 + \tfrac{2}{3}a_2.$$

b. More generally, for fixed n, let $a_1, a_2, \ldots, a_n \in \mathbf{R}$ be given and define the sequence $(a_k)_{k=1}^{\infty}$ by setting

$$a_k = \frac{a_{k-1} + a_{k-2} + \cdots + a_{k-n}}{n}$$

for $k \geqslant n+1$. Prove that $L(a_1, a_2, \ldots, a_n) \equiv \lim_{k\to\infty} a_k$ exists and that L is a linear function on $\mathbf{R}^n$. That is, show that there exist constants $c_j \in \mathbf{R}$ such that $L(a_1, a_2, \ldots, a_n) = c_1a_1 + c_2a_2 + \cdots + c_na_n$. Evaluate these constants c_j.

1265. *Proposed by M. S. Klamkin, University of Alberta, Canada.*

Determine the maximum area F of a triangle ABC if one side is of length λ and two of its medians intersect at right angles.

1266. *Proposed by John W. Goppelt, Haverford, Pennsylvania.*

a. Prove that every finite group is isomorphic to a group of even permutations.

b. Let G be a finite group and H a subgroup of index two in G. Prove that there is a group of permutations isomorphic to G whose even permutations correspond to H.

Quickies

Solutions to Quickies appear at the conclusion of the Problems section.

Q719. *From the 1986 U.S.A. Mathematical Olympiad.*

During a certain lecture, each of five mathematicians falls asleep exactly twice. For each pair of these mathematicians, there was some moment when both were sleeping simultaneously. Prove that, at some moment, some three of them were sleeping simultaneously.

Q720. *Proposed by John Sawka, Santa Clara University, Santa Clara, California.*

If A is a nonsingular $n \times n$ matrix, the adjoint of A, $\operatorname{adj} A$, can be expressed as $(\det A)A^{-1}$. Find a similar expression for $\operatorname{adj}(\operatorname{adj} A)$.

Q721. *Proposed by John P. Hoyt, Lancaster, Pennsylvania.*

Sketch the graph of $y = \cos x \cos(x+2) - \cos^2(x+1)$.

Solutions

A Line Segment Game **April 1986**

1237. *Proposed by David Gale, University of California, Berkeley.*

A two-person game is played in the following manner. Start with n points in the plane situated so that no three are collinear. Players take turns drawing line segments between these points. The only stipulation is that line segments are not permitted to intersect the interior of previously drawn line segments (they may share common endpoints). The first person who is unable to draw a line segment in this manner is the loser of the game. Show that the outcome of the game depends only on the given configuration of points and is independent of the strategies of the two players. Find a "formula" for determining the winner.

Solution by Joan Hutchinson and Stan Wagon, Smith College, Northampton, Massachusetts.

We will assume that $n \geq 3$, otherwise the problem is easily solved. Let P be the polygon that is the convex hull of the given points, and suppose k of the given points lie on P. When the game is over, all edges of P will be drawn, and P's interior will be divided into triangles. The latter follows from the fact that any polygon with more than three sides has a pair of nonadjacent vertices such that the line segment connecting them lies inside the polygon. (Note that this is false for polyhedra; for instructions on constructing a counterexample, called a *Lennes polyhedron*, see Howard Eves, *A Survey of Geometry*, vol. 1, Allyn and Bacon, 1964. This reference also contains a proof of the result for polygons.)

Given a triangulation of P we may compute the number of edges as follows. Choose any vertex on P and connect it, via P's exterior, to the $k-3$ other vertices of P not already adjacent to it. This results in a triangulation of $\mathbf{R}^2$ which, by Euler's formula, $V-E+F=2$, has E edges, where $E=n+(2/3)E-2$ (using $3F=2E$). Therefore, $E=3n-6$ and the number of edges when the game is over is $S \equiv 3n-6-(k-3)=3n-k-3$.

Now, independent of the way the game is played, the first player will win if and only if S is odd. The preceding formula for S shows that the outcome depends only upon n and k, invariants of the given configuration. We see from the formula that the first player wins if and only if n and k have the same parity, or equivalently, if and only if $n-k$, the number of points interior to P, is even.

The proof works for any set of n points in the plane, regardless of possible collinearity, provided it is stipulated that neither player can draw a line containing more than two of the given points.

Also solved by Kenneth L. Bernstein, Ada Booth, James Grochocinski, Jerrold W. Grossman, Erich Hauenstein (student), Thomas Jager, L. R. King and S. L. Davis and I. C. Bivens (jointly), David Kotz (student), Oxford Running Club (University of Mississippi), Ken Rebman, Iraj Saniee, Harry Sedinger, Gary Stevens, and the proposer. There was one incomplete solution and one incorrect solution.

A Congruent Incircle Point **April 1986**

1238. *Proposed by Clark Kimberling, University of Evansville.*

a. Prove that the interior of a triangle ABC contains a point P for which the three triangles APB, BPC, CPA have congruent incircles.

*b. Is P uniquely determined? Can the radii be determined? What can you say about the properties of P?

I. *Solution to part (a) by the proposer.*

For $r > 0$, let K_r be the circle of radius r centered at vertex A. Let s, t be the points where K_r crosses segments AB, AC, respectively. Let u be a point that moves on K_r between s and t, inside triangle ABC. Let j_u be the inradius of triangle AuB, and let k_u be the inradius of triangle AuC. Then

$$\lim_{u \to s} j_u = 0, \quad \lim_{u \to t} j_u > 0, \qquad \lim_{u \to s} k_u > 0, \qquad \lim_{u \to t} k_u = 0.$$

By the Intermediate Value Theorem, applied to the continuous function $j_u - k_u$, there exists a value u_r of u for which $j_u = k_u$. That is, Au_rB and Au_rC have equal radii.

Let R be the locus of u_r for $0 < r < \infty$. As u moves away from A, the radius of the congruent incircles of triangles Au_rB and Au_rC increases from 0, while that of triangle Bu_rC decreases to 0. Consequently, there must exist a value $r = r'$ for which these three circles have equal radii. That is, the point $u_{r'}$ is the point P whose existence was asserted.

II. *Solution by Noam Elkies (student), Harvard University.*

We prove that every positive ratio among the three inradii occurs for a unique P.

Let r_A, r_B, r_C be the inradii of triangles BPC, CPA, APB respectively. Consider the map ψ from the interior of triangle ABC to $\mathbf{RP}^2$ which takes P to the point with homogeneous coordinates (r_A, r_B, r_C). The range of ψ must clearly be in the portion of $\mathbf{RP}^2$ where all of the homogeneous coordinates can be taken to be positive. This is the interior of a projective triangle. Furthermore, as P traverses the boundary of triangle ABC, $\psi(P)$ goes once around the boundary of the projective triangle. But the interior of triangle ABC is contractible; thus so is its image under ψ. Therefore, this image contains every point with positive homogeneous coordinates, in particular $(1,1,1)$. This proves (a).

Now in the interior of triangle ABC, ψ is a local homeomorphism. (One way to see this: the directional derivative of r_A along **PA** is strictly positive, those of r_B, $r_C \leqslant 0$; similarly along **PB**, **PC**. So each tangent vector at $\psi(P)$ occurs as a directional derivative.) The range of ψ is simply connected, and, therefore, ψ is one-to-one. (The standard example ψ: $\mathbf{C}\setminus\{0\} \to \mathbf{C}$, $z \mapsto \ln z$, shows that simple connectedness is essential here.) This proves an extension of the uniqueness of part (b). The argument is nonconstructive, but applies to a much wider range of functions (with the inradius replaced by the area, semiperimeter, etc., or any positive combination thereof).

Irrational Diagonals — April 1986

1239. *Proposed by Bruce Hanson, St. Olaf College.*

Let $r_1, r_2, r_3, \ldots$ be any enumeration of the rationals in the interval $(0,1)$. For each $j \geqslant 1$, let $r_j = .a_{j,1}a_{j,2}a_{j,3}\ldots$ be a decimal representation of r_j.

a. Prove that the "main diagonal" $.a_{1,1}a_{2,2}a_{3,3}\ldots$ is irrational.

b. For arbitrary positive integers j and k, prove that the "diagonal" $.a_{j,k}a_{j+1,k+1}a_{j+2,k+2}\ldots$ is irrational.

Solution by Erwin Just, Bronx Community College.

It suffices to prove part (b). Assume to the contrary that the "diagonal" is rational. Define $t = .t_1t_2t_3\ldots$ in the following way. For $1 \leqslant i < k$, set $t_i = 0$, and for $i = 0, 1, 2, \ldots$, set $t_{k+i} = 5$ if $a_{j+1,k+i} \neq 5$, and $t_{k+i} = 4$ otherwise. Since the "diagonal" is rational, its decimal expansion is eventually periodic, and therefore by the manner in which t was constructed, t is also eventually periodic. Thus, t is a rational number in the interval $(0,1)$, and therefore it must appear in the enumeration. However, t cannot be among the rationals $r_j, r_{j+1}, r_{j+2}, \ldots$ because t and r_{j+i} differ in their $(k+i)$th coordinate. Thus t must be among $r_1, r_2, \ldots, r_{j-1}$. Now construct $v = .v_1v_2v_3\ldots$ so that $v_i \neq a_{i,i}$ and $v_i \neq a_{j-k+i,i}$ for $k \leqq i < j$, for $i < j$ and $v_i = t_i$ otherwise. Then v is a rational number in $(0,1)$ which is not found in the given enumeration. This contradiction proves that the "diagonal" is irrational.

Also solved by Kenneth L. Bernstein, Robert Berstein, A. K. Desai and Nimish Shah (India), David Dukelow, Noam D. Elkies (student), Alberto Facchini (Italy), Thomas Jager, N. J. Lord (England), Bill Mixon, G. Monroe (Canada), Howard Morris, Stephen Noltie, Gary E. Stevens, Douglas H. Underwood, University of North Carolina Problem-Solvers (Chapel Hill), William P. Wardlaw and Craig K. Bailey, Western Maryland College Problems Group, Gordon Williams, and the proposer.

Gary Stevens and *Erwin Just* showed that $d = .a_{\sigma(1),1}a_{\sigma(2),2}a_{\sigma(3),3}\ldots$ must be irrational for any permutation σ of the natural numbers. *Thomas Jager*, *Erwin Just*, and *Howard Morris* each pointed out that neither (a) nor (b) are necessarily true if the rationals are written in their binary representations. In this case, for example, it is possible to list the rationals so that 0.01111... appears as the main diagonal. The results are true, however, for any base $b > 2$. These results, and many others, can be found on microfilm (Xerox University Microfilms, Ann Arbor, Michigan) in the Ph.D. dissertation "On Properties of Diagonalizations of Certain Sequences" by Erwin Just. Also, see Problem #1262 in this issue.

An Infinite Series with Harmonic Numbers — April 1986

1240. *Proposed by Stephen Wayne Coffman (student), Western Maryland College.*

Evaluate

$$\sum_{n=1}^{\infty} \frac{H_n}{n \cdot 2^n}, \quad \text{where } H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

I. *Solution by Bruce Shawyer, Memorial University of Newfoundland, Canada.*

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \sum_{k=0}^{n-1} \frac{1}{k+1}.$$

Note that $f(0) = 0$ and that the radius of convergence is 1. Thus, for $|x| < 1$,

$$f'(x) = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n-1} \frac{1}{k+1} = \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{n=k+1}^{\infty} x^n$$

$$= \frac{1}{1-x} \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = \frac{-\log(1-x)}{x(1-x)}.$$

It follows that

$$f(1/2) = \int_0^{1/2} \frac{-\log(1-x)}{x(1-x)}\, dx = \int_0^1 \frac{\log(1+t)}{t}\, dt$$

(using the substitution $t = x/1-x$)

$$= -\int_0^1 \frac{1}{t} \sum_{n=0}^{\infty} \frac{(-t)^{n+1}}{n+1}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = \frac{\pi^2}{12}.$$

(The interchange of integral and sum in the last line is valid by an application of Abel's Limit Theorem.)

II. *Solution by Hans Kappus, Switzerland.*

The problem is a special case of the following formula, found as Exercise 20 in D. E. Knuth, *The Art of Computer Programming*, v. 1, Addison Wesley, 1977, p. 78.

If $f(x) = \sum_{n=1}^{\infty} a_n x^n$ converges, then

$$\sum_{n=1}^{\infty} a_n H_n x^n = \int_0^1 \frac{f(x) - f(tx)}{1-t}\, dt.$$

In this case, take $a_n = 1/n$, so that $f(x) = -\log(1-x)$, and we have, for $x = 1/2$,

$$\sum_{n=1}^{\infty} \frac{H_n}{n \cdot 2^n} = \int_0^1 \frac{\log(2-t)}{1-t}\, dt = \int_0^1 \frac{\log(1+t)}{t}\, dt = \pi^2/12.$$

III. *Solution by Bruce C. Berndt, University of Illinois, Urbana.*

The result is easily deduced from more general considerations of Ramanujan. References below are to B. C. Berndt, *Ramanujan's Notebooks*, Part I, Springer-Verlag, New York, 1985.

Write

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n\cdot 2^n} &= \sum_{n=1}^{\infty}\frac{H_{n-1}+1/n}{n\cdot 2^n}, \qquad (H_0=0)\\ &= \sum_{n=1}^{\infty}\frac{H_n}{(n+1)2^{n+1}}+\sum_{n=1}^{\infty}\frac{1}{n^2\cdot 2^n}\\ &= \frac{1}{2}\log^2 2+\left(\frac{\pi^2}{12}-\frac{1}{2}\log^2 2\right)\\ &= \pi^2/12,\end{aligned}$$

by (9.2), p. 251, and Example (i), p. 248.

Also solved by J. Marshall Ash, Kenneth L. Bernstein, Paul Bracken, Chico Problem Group, L. Matthew Christophe, Jr., Noam D. Elkies, Leslie V. Glickman, James Grochocinski, H. Vichier-Guerre (student), Douglas Henkin (student), Thomas Jager, L. Kuipers (Switzerland), Kee-wai Lau (Hong Kong), N. J. Lord (England), David E. Manes, Roger H. Moritz, Howard Morris, Roger B. Nelson, David E. Penney, I. Peters, Ranjan Roy, Edward Schmeichel, Robert E. Shafer, Heinz-Jürgen Seiffert (West Germany), Byron Siu, Michiel Smid (The Netherlands), J. M. Stark, Michael Vowe (Switzerland), Yan-loi Wong (student), Staffan Wrigge (Sweden), Morton Zweiback, and the proposer.

The *Chico Problem Group* provided several generalizations and cited the following references: L. Lewin, *Polylogarithms and Associated Functions*, North Holland, New York, 1981, and an unpublished manuscript "Certain Integrals and Series Related by the Zeta Function", by Michael J. Dixon (California State University, Chico) and C. O'Cinneide.

An Isometry on a Compact Metric Space — April 1986

1241. *Proposed by Richard Johnsonbaugh, DePaul University, and Sadahiro Saeki, Kansas State University.*

Does there exist a compact metric space M with an isometry that is into, but not onto, M?

Solution by Edwin M. Klein, University of Wisconsin, Whitewater.

No. Let f be an isometry of M into itself, let $a \in M$, and let $\varepsilon > 0$. The sequence $(f^{(n)}(a))$ has a Cauchy subsequence, so there exist positive integers $n > m$ such that $d(f^{(n)}(a), f^{(m)}(a)) < \varepsilon$. Thus, $(f^{(n-m)}(a), a) = d(f^{(n)}(a), f^{(m)}(a)) < \varepsilon$. It follows that $a \in \overline{f(M)} = f(M)$.

Also solved by A. K. Desai (India), Patrick R. Gardner, Douglas Henkin (student), Thomas Jager, Erwin Kronheimer (England), David Kotz (student), N. J. Lord (England), David E. Manes, Kee-wai Lau (Hong Kong), Stephen Noltie, Oxford Running Club (University of Mississippi), Gordon Williams, Yan-loi Wong (student), and the proposers.

Several people noted that the problem appears as an exercise in many standard topology textbooks; e.g. as Exercise D, p. 162, in J. L. Kelley's *General Topology* (D. Van Nostrand, New York, 1955).

Answers

Solutions to the Quickies which appear near the beginning of the Problems section.

A719. The following is a paraphrasing of the brilliant solution given by Joseph G. Keane (Pittsburgh, Pa.) on the 1986 U.S.A. Olympiad.

If this were not the case, there must be $\binom{5}{2} = 10$ disjoint intervals of time when exactly two mathematicians are asleep. The initial point of each of these intervals is a time when a mathematician has just dozed off. This accounts for all the "dozing off" times for the five mathematicians because each falls asleep exactly twice. Now consider the first instant when two mathematicians are asleep. This is a dozing-off time for one mathematician, but some other mathematician had to have dozed off at or before this time. This is impossible because all ten dozing-off times are already identified with intervals.

A720. Observe that $\det(\operatorname{adj} A) = (\det A)^{n-1}$, and it follows that $\operatorname{adj}(\operatorname{adj} A) = \det(\operatorname{adj} A)(\operatorname{adj} A)^{-1} = (\det A)^{n-1}(\det A)^{-1}A = (\det A)^{n-2}A$.

A721. The derivative of $y = \cos x \cos(x+2) - \cos^2(x+1)$ is

$$\begin{aligned}\frac{dy}{dx} &= -\sin x \cos(x+2) - \cos x \sin(x+2) + 2\cos(x+1)\sin(x+1)\\ &= -\sin(x+x+2) + \sin(2(x+1)) = 0,\end{aligned}$$

and, therefore, the graph is a horizontal line through the point $(\pi/2, -\sin^2 1)$.

REVIEWS

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Davis, Philip J., and Reuben Hersh, *Descartes' Dream: The World According to Mathematics*, Harcourt Brace Jovanovich, 1986; xviii + 321 pp, $19.95.

Already selected for one book club, this is likely to be the one-book-in-five-years that our cultured and curious nonmathematical colleagues will read. Unfortunately, it will tend to reinforce any misgivings they have about the humanist value of mathematics and computing, confirm any confusion of the two, and serve to justify math avoidance and mathophobia. These undesirable consequences flow not from a bad book--on the contrary, the writing and elegant arguing are absolutely first-rate--but from the authors' philosophical disposition. Their antiplatonist stand denies the possibility of "timeless truths" in mathematics and rejects as harmful a perceived increasing "mathematization" of the world. The book ends with a challenge to "mathematics-computers" to become a "human institution": that is, society must develop "self-awareness that in its ordinary mathematical usages it is arranging itself in certain ways and hence is doing something to itself." Mathematicians and computer scientists have tended to practice their art in private, acting neutral toward the appplications to which their work is put. Reuben and Hersh want us not only to wake up to our profession's complicity in building toward nuclear holocaust, but also to realize that anything that we create that *can* be used to the detriment of human beings *will* be--and we can no longer pretend innocent ignorance. (The physicist of p. 158 should be Hawking, there's some text missing on p. 161, and Lobachevsky's name comes by inconsistent spelling.)

Crutchfield, James P., et al., Chaos, *Scientific American* 255:6 (December 1986) 46-57, 152.

Excellent introduction to the physical basis and qualitative aspects of chaos, relying on the reader's intuition of state space. But more than just an exposition, this article comments on the philosophical consequences of the discovery of simple deterministic systems with random behavior: "The existence of chaos affects the scientific method itself...long-term predictions are intrinsically impossible...the hope that physics could be complete with an increasingly detailed understanding of fundamental physical forces and constituents is unfounded. The interaction of components on one scale can lead to complex global behavior on a larger scale that in general cannot be deduced from knowledge of the individual components." And yet it may be that "chaos provides a mechanism that allows for free will within a world governed by deterministic laws."

Fischer, Gerd, *Mathematische Modelle/Mathematical Models* and *Mathematical Models Commentary*, Vieweg, 1986; xii + 132 pp and viii + 83 pp, DM 118 (distributed in the U.S. by International Publishers Service, $52.00.)

A book of photographs and a book of mathematical explanations of classic mathematical models--models in the original plaster sense!--gathered from collections in universities and museums. Most date from the 1870s and 1880s; they were expensive then and are rare now. Many were constructed under the direction of Felix Klein. This boxed pair of books performs a real service both for those whose departments (like mine) have mourned the lack of documentation, and for those who will now marvel at these creations for the first time. But more wondrous still, the publishers also offer careful reproductions of four of the models, in gypsum and in the original size, at prices ranging from DM 98 to DM 198. (Note: The *Commentary* volume is available also in German.)

Berndt, Bruce C., *Ramanujan's Notebooks, Part I*, Springer-Verlag, 1985; 357 pp, $54.00.

"This volume is the first of three volumes devoted to the editing of Ramanujan's notebooks. Many of the results found herein are very well known, but many are new ... Our goal has been to prove each of Ramanujan's Theorems." Author Berndt reformulates many ambiguous results, supplying adequate hypotheses, precise statements, and rigorous proofs. Now, at last, there is a convenient source from which to enjoy Ramanujan's legacy.

Steinhardt, Paul Joseph, Quasicrystals, *American Scientist* 74 (November-December 1986) cover, 586-597.

"Quasiperiodic structures contain two or more shapes used over and over again in a predictable but subtle sequence that never quite repeats." In 1974 Roger Penrose, on a recreational mathematics lark, invented the first quasiperiodic tilings of the plane. Recently, theoretical physicists have hypothesized a new phase of solid matter, neither crystalline nor glassy, with atoms arranged in a three-dimensional quasiperiodic array. Coincidentally, experimental physicists were creating puzzling new alloys that appear to have quasiperiodic structure. You can read more about quasicrystals in *Scientific American* 255:2 (August 1986) 42-51, 120, and about aperiodic patterns in Ch. 10 of Grünbaum and Shephard's great *Tilings and Patterns* (Freeman, 1987).

Howson, A. G., and J. P. Kahane, *The Influence of Computers and Information on Mathematics and Its Teaching*, Cambridge University Press, 1986; vii + 155 pp., $39.50.

Apart from papers exploring details of the new symbiosis between computers and mathematics teaching, there are a couple of real classics here: J. Davenport on the mathematics of computer algebra (with its news that typical tables of integrals contain 10-25% errors!), and L. A. Steen's inimitable "Living with a new mathematical species" (with its eloquent damning of computer literacy courses, and its "very old answer" to the "new question" about "the wisdom of teaching skills [such as differentiation] that computers can do as well or better than humans.")

Weizenbaum, Joseph, Not without us, *Fellowship* (October/November 1986) 8-10.

"None of the weapons that today threaten every human being with murder, and whose design, manufacture and sale condemns countless people to starvation, could be developed without the earnest cooperation of computer professionals...Today we know with virtual certainty that every scientific and technical result will, if at all possible, be put to use in military systems...[w]e have the power either to increase the efficiency of the mass murder instruments and thereby make the murder of our children more likely, or to bring the present insanity to a halt...Let us think about what we accomplish in our work, about how it will be used, and whether we are in the service of life or death."

Grünbaum, Branko, and G. C. Shephard, *Tilings and Patterns*, Freeman, 1987; ix + 700 pp, $59.95.

Long, long awaited, with preliminary drafts available ten years ago, this book is *the* permanent authority on planar tilings and patterns of all kinds. Most of the exposition is of work by the authors, who have exercised great discernment in defining and naming the basic concepts of the subject. Exercises are plentiful and open problems abound. Why not consider this volume for a senior or faculty seminar? The rewards are well worth it.

NEWS & LETTERS

47th PUTNAM COMPETITON: WINNERS AND SOLUTIONS

Teams from 270 schools competed in the 1986 William Lowell Putnam mathematical competition. The top five winning teams, in descending rank, are:

Harvard University
Douglas S. Jungreis, Bjorn M. Poonen, David J. Zuckerman

Washington University, St. Louis
Daniel N. Ropp, Dougin A. Walker, Japheth L.M. Wood

University of California, Berkeley
Michael J. McGrath, David P. Moulton, Christopher S. Welty

Yale University
Thomas O. Andrews, Kamal F. Khuri-Makdisi, David R. Steinsaltz

Massachusetts Institute of Technology
David Blackston, Jim P. Ferry, Waldemar P. Horwat

The six highest ranking individuals, named Putnam Fellows, are:

David J. Grabiner	Princeton Univ.
Waldemar P. Horwat	MIT
Douglas S. Jungreis	Harvard University
David J. Moews	Harvard University
Bjorn M. Poonen	Harvard University
David J. Zuckerman	Harvard University

Solutions to the 1986 *Putnam problems were prepared for publication in this* Magazine *by Loren Larson, St. Olaf College.*

A-1. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

Sol. The condition that $x^4 + 36 \leq 13x^2$ is equivalent to $(x-3)(x-2)(x+2)(x+3) \leq 0$. The latter is satisfied if and only if x is in the closed interval $[-3,-2]$ or the closed interval $[2,3]$. The function f is increasing on these intervals because for such x, $f'(x) = 3(x^2-1) > 0$. It follows that the maximum value of f over this domain is $\max\{f(-2),f(3)\} = 18$.

A-2. What is the units (i.e., rightmost) digit of $\left[\dfrac{10^{20000}}{10^{100}+3}\right]$? Here $[x]$ is the greatest integer $\leq x$.

Sol. Let $a = 10^{100} + 3$. Then

$$\left[\frac{10^{20000}}{10^{100}+3}\right] = \left[\frac{(a-3)^{200}}{a}\right] = \left[\frac{1}{a}\sum_{k=0}^{200}\binom{200}{k}a^{200-k}(-3)^k\right] = \sum_{k=0}^{199}\binom{200}{k}a^{199-k}(-3)^k$$

Now, modulo 10,

$$\sum_{k=0}^{199}\binom{200}{k}a^{199-k}(-3)^k \equiv \sum_{k=0}^{199}\binom{200}{k}3^{199-k}(-3)^k \equiv 3^{199}\sum_{k=0}^{199}(-1)^k\binom{200}{k}.$$

But $\sum_{k=0}^{200}(-1)^k\binom{200}{k} = 0$, so the last expression is $\equiv 3^{199}\left(0 - \binom{200}{200}\right) \equiv 3$.

A-3. Evaluate $\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2+n+1)$, where Arccot t for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot\theta = t$.

Sol. Let S_n denote the nth partial sum. Then $S_1 = \operatorname{Arccot} 1 + \operatorname{Arccot} 3$. Using $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$, one sees that $\cot S_1 = 1/2$, and therefore $S_1 = \operatorname{Arccot}(1/2)$. In the same way one shows by induction that $S_n = \operatorname{Arccot}\left(\frac{1}{n+1}\right)$. Thus the series sums to $\lim_{n\to\infty} S_n = \operatorname{Arccot} 0 = \pi/2$.

A-4. A *transversal* of an $n\times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n\times n$ matrices A satisfying the following conditions:

(a) Each entry of A is either -1, 0, or 1;

(b) The sum of the n entries of a transversal is the same for all transversals of A.

An example of such a matrix A is

$$A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1b_1^n + a_2b_2^n + a_3b_3^n + a_4 ,$$

where the a_i's and b_i's are rational numbers.

Sol. We first prove:

Lemma. An $n\times n$ matrix (α_{ij}) satisfies (b) if and only if

$$(*) \quad \alpha_{i1}-\alpha_{h1} = \alpha_{i2}-\alpha_{h2} = \cdots = \alpha_{in}-\alpha_{hn}$$

for each i and h.

Proof of Lemma: Assume that (α_{ij}) satisfies (b). Let $i \neq h$ and $j \neq k$. Both α_{ij} and α_{hk} appear in some transversal T. Replacing these two entries with α_{ik} and α_{hj} gives us another transversal T'. Using (b), we see that $\alpha_{ij} + \alpha_{hk} = \alpha_{ik} + \alpha_{hj}$, which implies that $\alpha_{ij} - \alpha_{hj} = \alpha_{ik} - \alpha_{hk}$. This is also true if $i = h$ or $j = k$. Thus we have shown that (b) implies (*).

Conversely, assume (*) for all i and h. A transversal consists of entries $\alpha_{1j_1}, \alpha_{2j_2}, \ldots, \alpha_{nj_n}$ with $j_1, j_2, \ldots, j_n$ a permutation of $1,2,\ldots,n$. Such a permutation can be changed into any other permutation (and the transversal changed into any other transversal) by a finite sequence of interchanges of j_i and j_h. Thus (*) implies (b).

Now we use the lemma and a consideration of the possibilities for the first row of (α_{ij}) to find $f(n)$. For a given matrix, let S consist of the integers $\{-1,0,1\}$ that actually appear on the first row. If $\#S = 1$ (that is, $S = \{-1\}$, or $\{0\}$ or $\{1\}$), it follows from the lemma that each row has equal entries. There are three such first rows and 3^n such matrices since there are 3 choices for each row. If $S = \{0,1\}$, the entries on the ith row are either $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}$ or $\alpha_{11}-1, \alpha_{12}-1, \ldots, \alpha_{1n}-1$. There are 2^n-2 such first rows and then 2^{n-1} choices for the other rows for a total of $2^{n-1}(2^n-2)$ such matrices. The same count is obtained when $S = \{-1,0\}$. In the remaining cases (that is, those with $S = \{1,-1\}$, or $S = \{-1,0,1\}$) the lemma implies that all rows are the same. The number of such first rows, and hence of such matrices, is $3^n - 2(2^n-2) - 3 = 3^n - 2\cdot 2^n + 1$. Since our classification is exhaustive and nonoverlapping,

$$\begin{aligned} f(n) &= [3^n - 2\cdot 2^n + 1] + 2[2^{n-1}(2^n-2)] + 3^n \\ &= 4^n + 2\cdot 3^n - 4\cdot 2^n + 1 . \end{aligned}$$

A-5. Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are functions of n real variables $x = (x_1,\ldots,x_n)$ with continuous second order partial derivatives everywhere on R^n. Suppose further that there are constants c_{ij} such that $\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$ for all i and j, $1 \leq i \leq n$, $1 \leq j \leq n$. Prove that there is a function $g(x)$ on R^n such that $f_i + \frac{\partial g}{\partial x_i}$ is linear for all i, $1 \leq i \leq n$. (A linear function is one of the form $a_0 + a_1x_1 + a_2x_2 + \ldots + a_nx_n$.)

Sol. Note that $c_{ji} = -c_{ij}$ for all i and j. Let $h_i = \frac{1}{2}\sum_j c_{ij}x_j$ so that $\frac{\partial h_i}{\partial x_j} = \frac{1}{2}c_{ij}$. Then $\frac{\partial h_i}{\partial x_j} - \frac{\partial h_j}{\partial x_i} = \frac{1}{2}c_{ij} - \frac{1}{2}c_{ji} = c_{ij} = \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}$ and so $\frac{\partial(h_i - f_i)}{\partial x_j} = \frac{\partial(h_j - f_j)}{\partial x_i}$ for all i and j. Hence $(h_1-f_1, \ldots, h_n-f_n)$ is a gradient and so there is a function g such that $\frac{\partial g}{\partial x_i} = h_i - f_i$. In other words, $f_i + \frac{\partial g}{\partial x_i} = h_i$ is linear.

A-6. Let $a_1,a_2,\ldots,a_n$ be real numbers, and let $b_1,b_2,\ldots,b_n$ be distinct positive integers. Suppose there is a polynomial $f(x)$ satisfying the identity

$$(1 - x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i} .$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1,b_2,\ldots,b_n$ and n (but independent from $a_1,a_2,\ldots,a_n$).

Sol. Write $(b)_j = b(b-1)\cdots(b-j+1)$.

Differentiating $(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}$ j times $(0 \leq j \leq n)$ and putting $x = 1$ yields

$$\begin{aligned} 0 &= 1 + \sum a_i, \\ 0 &= \sum a_i b_i, \\ 0 &= \sum a_i (b_i)_2, \\ &\vdots \\ 0 &= \sum a_i (b_i)_{n-1}, \\ n!f(1) &= \sum a_i (b_i)_n. \end{aligned}$$

Solve the first n equations for $a_1, \ldots, a_n$ by Cramer's rule and substitute into the last equation. We get $n!f(1) =$

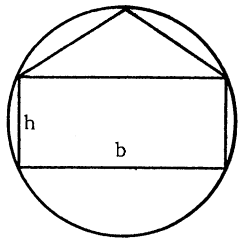
$$\frac{\sum_{i=1}^{n} (b_i)_n (-1)^i \begin{vmatrix} b_1 & \cdots & \widehat{b_i} & \cdots & b_n \\ & & & & \\ (b_1)_{n-1} & \cdots & \widehat{(b_i)}_{n-1} & \cdots & (b_n)_{n-1} \end{vmatrix}}{\begin{vmatrix} 1 & & 1 & & 1 \\ b_1 & \cdots & b_2 & \cdots & b_n \\ & & \vdots & & \\ (b_1)_{n-1} & \cdots & (b_2)_{n-1} & \cdots & (b_n)_{n-1} \end{vmatrix}}$$

where $\wedge$ indicates a missing entry. The denominator D is a polynomial of total degree $\binom{n}{2}$ in $b_1, \ldots, b_n$ and vanishes whenever $b_i = b_j$, $i \neq j$. Hence $D = C \prod_{i<j} (b_i - b_j)$. The constant C is seen to be 1 by considering, say, the coefficient of $b_1 b_3^2 \cdots b_n^{n-1}$, so $D = \prod_{i<j} (b_i - b_j)$. (This can also be proved by performing elementary row operations on the Vandermonde determinant.)

Put $b_i = b_j$ $(i \neq j)$ in the numerator. All terms vanish except two, which are of equal magnitude but opposite sign. Hence the numerator is divisible by $b_i - b_j$ and thus by the denominator. Therefore $n!f(1)$ is a polynomial in $b_1, \ldots, b_n$. The degree of this polynomial is $\leq n$, since each term in the numerator has degree n more than the degree $\binom{n}{2}$ of the denominator.

Now put $b_i = 0$. The denominator doesn't vanish, but each term in the numerator does. Thus, $n!f(1)$ is divisible by b_i, so $n!f(1) = kb_1 b_2 \ldots b_n$ for some constant k. By putting $f(x) = 1$ (so $b_i = i$), we see $k = 1$. Thus $f(1) = b_1 \ldots b_n / n!$.

B-1. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown.



For what value of h do the rectangle and triangle have the same area?

Sol. The altitude of the triangle is $\frac{1}{2}(2-h)$. Equal area means $h = \frac{1}{2}$ (altitude of triangle) $= \frac{1}{4}(2-h)$, so $h = 2/5$.

B-2. Prove that there are only a finite number of possibilities for the ordered triple $T = (x-y, y-z, z-x)$ where x, y, and z are complex numbers satisfying the simultaneous equations $x(x-1)+2yz = y(y-1)+2zx = z(z-1)+2xy$, and list all such triples T.

Sol. Let $a = x - y$ and $b = y - z$ (so $z - x = -(a + b)$). Then $x = a + y$ and $z = y - b$, and substituting these into the equation (to reduce each to terms of y) we find that the equations are equivalent to

$$a^2 - a = -2ab = b^2 + b.$$

If $a \neq 0$, then $b = \frac{1-a}{2}$, and substituting this into $a^2 - a = b^2 + b$ gives $a = 1$ or $a = -1$. This yields the triples $(1,0,-1)$ and $(-1,1,0)$.

If $a = 0$, then $b^2 + b = 0$, so $b = 0$ or $b = -1$. This yields the triples $(0,0,0)$ and $(0,-1,1)$.

Thus, there are only four possibilities, and each of these occur.

B-3. Let Γ consist of all polynomials in x with integer coefficients. For f and g in Γ and m a positive integer, let $f \equiv g \pmod{m}$ mean that every coefficient of $f - g$ is an integral multiple of m.

Let n and p be positive integers with p prime. Given that f, g, h, r, and s are in Γ with $rf + sg \equiv 1 \pmod p$ and $fg \equiv h \pmod p$, prove that there exist F and G in Γ with $F \equiv f \pmod p$, $G \equiv g \pmod p$, and $FG \equiv h \pmod{p^n}$.

Sol. Suppose for $k \geq 1$ that we have polynomials F_k and G_k with integer coefficients such that $F_k \equiv f \pmod p$, $G_k \equiv g \pmod p$, and $F_kG_k - h \equiv 0 \pmod{p^k}$. For $k = 1$ this can be done with $F_1 = f$, $G_1 = g$.

From these congruences and $rf + sg \equiv 1 \pmod p$ we get $rF_k + sG_k - 1 \equiv 0 \pmod p$. Hence

$$(h - F_kG_k)(rF_k + sG_k - 1) \equiv 0 \pmod{p^{k+1}},$$

or equivalently

$$F_kG_k + F_kr(h-F_kG_k) + G_ks(h-F_kG_k) \equiv h \pmod{p^{k+1}}$$

Also, from $h - F_kG_k \equiv 0 \pmod{p^k}$, we have $rs(h - F_kG_k)^2 \equiv 0 \pmod{p^{k+1}}$. Adding this to the last congruence we have

$$(F_k + s(h-F_kG_k))(G_k + r(h-F_kG_k)) \equiv h \pmod{p^{k+1}}.$$

Let $F_{k+1} = F_k + s(h - F_kG_k)$, $G_{k+1} = G_k + r(h - F_kG_k)$. Then $F_{k+1}G_{k+1} \equiv h \pmod{p^{k+1}}$, $F_{k+1} \equiv F_k \equiv f \pmod p$, $G_{k+1} \equiv G_k \equiv g \pmod p$ and by induction the proof is complete.

B-4. For a positive real number r, let $G(r)$ be the minimum value of

$\left|r - \sqrt{m^2 + 2n^2}\right|$ for all integers m and n. Prove or disprove the assertion that $\lim_{r\to\infty} G(r)$ exists and equals 0.

Sol. Let m be the largest integer in $N = \{0,1,\ldots\}$ with $r^2 \geq m^2$. Let n be the largest integer in N with $\frac{r^2 - m^2}{2} \geq n^2$. It follows that $r^2 - m^2 < 2m + 1 \leq 2r + 1$ and that $\frac{r^2 - m^2}{2} - n^2 < 2n + 1 \leq 2\sqrt{\frac{r^2 - m^2}{2}} + 1 < 2\sqrt{\frac{2r + 1}{2}} + 1 = \sqrt{2(2r + 1)} + 1$. Hence $r^2 - m^2 - 2n^2 < 2\sqrt{2}\sqrt{2r+1} + 2$. Since $r^2 - m^2 - 2n^2 = \left(r - \sqrt{m^2+2n^2}\right)\left(r + \sqrt{m^2+2n^2}\right)$,

$$r - \sqrt{m^2+2n^2} = \frac{r^2 - m^2 - 2n^2}{r + \sqrt{m^2+2n^2}} < \frac{2\sqrt{2}\sqrt{2r+1} + 2}{r},$$

which $\to 0$ as $r \to \infty$.

Hence $\lim_{r\to\infty} G(r) = \lim_{r\to\infty} \left|r - \sqrt{m^2+2n^2}\right| = 0.$

B-5. Let $f(x,y,z) = x^2 + y^2 + z^2 + xyz$. Let $p(x,y,z)$, $q(x,y,z)$, $r(x,y,z)$ be polynomials with real coefficients satisfying

$f(p(x,y,z), q(x,y,z), r(x,y,z)) = f(x,y,z)$. Prove or disprove the assertion that the sequence p, q, r consists of some permutation of $\pm x$, $\pm y$, $\pm z$, where the number of minus signs is 0 or 2.

Sol. The assertion is false. Take $p = x$, $q = y$, $r = -z - xy$.

B-6. Suppose A, B, C, D are $n\times n$ matrices with entries in a field F, satisfying the conditions that AB^t and CD^t are symmetric and $AD^t - BC^t = I$. Here I is the $n\times n$ identity matrix, and if M is an $n\times n$ matrix, M^t is the transpose of M. Prove that $A^tD - C^tB = I$.

Sol. The conditions of the problem are

(i) $AB^t = (AB^t)^t = BA^t$,
(ii) $CD^t = (CD^t)^t = DC^t$,
(iii) $AD^t - BC^t = I$.

Condition (i) implies $BA^t - AB^t = 0$ (the $n\times n$ zero matrix). Condition (ii) implies $CD^t - DC^t = 0$, and the transpose of condition (iii) is $DA^t - CB^t = I^t = I$. Hence we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} = \begin{pmatrix} AD^t-BC^t & -AB^t+BA^t \\ CD^t-DC^t & -CB^t+DA^t \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

From this it follows that

$$\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

as well, and the lower right-hand corner of this is $-C^tB + A^tD = I$.

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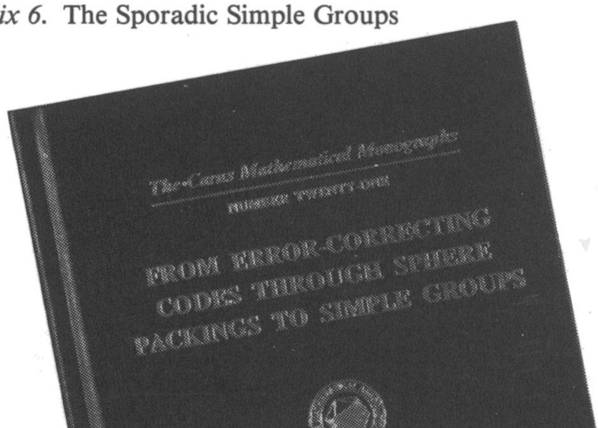

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